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ABSTRACT

The papers presented in these proceedings are organized into seven categories: (1) number and number operations (13 papers); (2) spatial representation and geometrical understanding (10 papers); (3) developing and/or using models of mathematical learning (15 papers); (4) mathematical concept formation (17 papers); (5) the mathematical learning environment (9 papers); (6) logic and proof (5 papers); and (7) problem solving strategies (13 papers). (RH)

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Psychology of Mathematics Education

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PREFACE

The 1986 London Conference of PME marks the tenth anniversary of the formation of the International Group for the Psychology of Mathematics Education at the third ICME in Karlsruhe. During that time, the group has grown considerably and the annual conferences have proved successful in their declared aim of fostering international communication about psychological aspects of problems of mathematics education at all levels. The range of headings under which the research reports contained in this volume have been organised attests to the flourishing of this area, and part of the success of PME has been in carving out a domain of enquiry within the emerging discipline of mathematics education.

The tenth annual conference takes place at the City University, London from July 20th to July 25th, 1986. There are a number of different ways in which participants at the conference may make a contribution; research reports, poster displays, working groups (initiated in 1984) and discussion groups (new this year). The background papers for the research reports, which are intended to be read before the presentation, form the bulk of the contents of this volume and have been organised under the following 7 headings:

1. Number and number operations
2. Spatial representation and geometrical understanding
3. Developing and/or using models of mathematical learning
4. Mathematical concept formation
5. The mathematical learning environment
6. Logic and proof
7. Problem solving strategies

The order in which they appear in this volume is alphabetic within each heading and therefore does not necessarily reflect the order of presentation within the meeting itself. The plenary sessions have been chosen to reflect diverse themes of potential interest to psychologists of mathematics education.

Any particular paper can be located by consulting either the table of contents at the beginning or the alphabetical list of contributors at the end. We would like to thank Nicolas Balacheff, Alan Bell, Margaret Brown and Joop van Dormolen for their assistance on the programme committee as part of the preparations for this conference. We are also particularly grateful to Rita Pryer of Thames Polytechnic for the immense ammount of administrative work involved in organising a conference of this size.

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1. NUMBER AND NUMBER OPERATIONS

FIRST-GRADERS' LEVELS OF VERBAL PROBLEM PERFORMANCE

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Abstract

This study used past findings on children's informal concrete modeling strategies for verbal problems and the resultant proposed models of performance to categorize 45 first-graders into three concrete representational levels. The children were given an instructional treatment intended to teach them to write formal symbolic representations that corresponded to their informal concrete representations. Preinstructional and postinstructional measures of children's performance, on verbal problem tasks at each level are stated in the results.

Young children's informal number concepts and successful strategies for solving verbal addition and subtraction problems have been documented. (See summaries in Carpenter, Blume, Hiebert, Auick, & Pimm, 1982; Ginsburg, 1983; Starkey & Gelman, 1982.) An important feature of children's earliest solution strategies is their understanding of problem structure, as evidenced by their attempts to represent problem structure with concrete items (Blume, 1981; Carpenter, Hiebert, & Moser, 1981; Carpenter & Moser, 1982, 1983, 1984; Hiebert, 1982). Before formal instruction, children set out concrete models that represent the underlying mathematical structure of simple verbal problems land then manipulate these concrete representations to find the solution.

Children come to formal instruction at different levels of verbal problem solving performance. Some children can solve only those problems with less apparent structures. Their progressively sophisticated strategies have been studied and hierarchical models of performance have been proposed by Briars and Larkin (1984), Carpenter and Moser (1983, 1984), Cobb (1986), and Riley, Greeno, and Heller (1983).

The present study used these previously proposed models to categorize children into three levels based on children's use of problem structure for their concrete representations. This categorization preceded a whole-class instructional treatment that taught the same mathematical content to children at all levels. This content intended to move children from their informal concrete representational strategies to the formal symbolic representational forms, i.e., number sentences or simple equations. The major question posed involved the efficacy of teaching structure-based symbolic representations to children at all three entering concrete representational levels.

Method

Sample

The subjects were 45 first-graders in two classrooms during the last quarter of the school year. The school was located in a rural community near Madison, Wisconsin.

Procedure

The children's performances were evaluated with individual interviews and group tests using problems from the current classification of verbal problems (Carpenter & Moser, 1984; Riley, et al., 1983). According to their use of concrete representational strategies during the individual interviews, children were categorized into the levels of Basic, Direct Modeling, or Representation. A group paper-and-pencil test was given to examine children's attempts to symbolically represent and solve verbal problems prior to instruction. Fourteen instructional lessons followed, with the intent of teaching children to symbolically represent verbal problems with number sentences that directly represented problem structure. A post-instructional group paper-and-pencil test of verbal problems was administered to determine the change in children's symbolic representations and solutions.

Results and Discussion

Children's preinstructional and postinstructional symbolic representations and solutions for Change type problems are presented in Tables 1, 2, and 3.

The study indicated several results. One major result was that after instruction in symbolically representing the structure of verbal problems, children categorized at all three concrete modeling performance levels were successful in writing correct number sentences for the instructed verbal problem types. The children had been categorized into levels prior to instruction in anticipation of the instructional treatment not being appropriate for all levels of first-graders. Posttest results indicated that over 80% of the children at all three concrete modeling levels were successful in writing correct number sentences for the instructed verbal problem types, Change and Combine problems.

Another result concerned the noninstructed problems, Compare and

Equalize problems. The instruction that children received in writing sentences to represent verbal problem structure did not appear to transfer to these two noninstructed problem types. Although over one-half of the children were able to solve these problem types on the posttest, less than 10% of the children wrote correct number sentences for Compare and Equalize problems.

The study indicated that young children at varying levels of preinstructional ability were successful in learning to symbolically represent and solve verbal problems with number sentences that directly represent verbal problem structure. The study suggests an early program of verbal problem solving instruction that teaches children the formal symbols of mathematics to coincide with their informal concrete modeling strategies.

Acknowledgements

The research reported in the study was performed for a doctoral dissertation with the supervision of T. P. Carpenter. The research was supported by Professor Carpenter's grant from the National Institute of Education (Grant No. NIE-G-84-0008) and by the Association of American Publishers. The opinions expressed in this paper do not necessarily reflect the position, policy, or endorsement of the National Institute of Education nor the Association of American Publishers.

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Table Preinstructional and Postinstructional Performance on Group Tests for Direct Modeling Level Students (N=22) - Change Problems

Verbal Problem Type		Correct Sentence	Correct Solution	Direct Representation			Rerepresentation			Other
				Form	Complete	Incomplete	Form	Complete	Incomplete	
Change 1	pre	16	15	8+7=□	16	3				3
	post	22	22	9+4=□	22	-				-
Change 2	pre	16	14	13-9=□	16	3				3
	post	20	19	12-9=□	20	2				-
Change 3	pre	1	7	9+□=12	--	4	12-9=□	1	-	17
	post	19	21	7+□=11	19	1	11-7=□	-	-	2
Change 4	pre	9	11	14-□=9	--	2	14-9=□	9	6	5
	post	21	16	15-□=8	16	-	15-8=□	5	-	1
Change 5	pre	0	4	□+6=13	--	2	13-6=□	-	-	20
	post	20	16	□+5=11	20	1	11-5=□	-	-	1
Change 6	pre	3	6	□-4=8	--	2	4+8=□	3	-	17
	post	20	20	□-5=8	20	-	5+8=□	-	-	2

Table Preinstructional and Postinstructional Performance on Group Tests for Basic Level Students (N=12) - Change Problems

Verbal Problem Type		Correct Sentence	Correct Solution	Direct Representation			Rerepresentation			Other
				Form	Complete	Incomplete	Form	Complete	Incomplete	
Change 1	pre	7	7	8+7=□	7	3				2
	post	11	8	9+4=□	11	-				1
Change 2	pre	9	5	13-9=□	9	2				1
	post	11	10	12-9=□	11	1				-
Change 3	pre	0	1	9+□=12	-	1	12-9=□	-	-	11
	post	9	7	7+□=11	9	-	11-7=□	-	-	3
Change 4	pre	7	8	14-□=9	-	1	14-9=□	7	2	2
	post	11	7	15-□=8	9	-	15-8=□	3	-	-
Change 5	pre	0	1	□+6=13	-	2	13-6=□	-	-	10
	post	10	10	□+5=11	10	-	11-5=□	-	-	2
Change 6	pre	4	6	□-4=8	-	1	4+8=□	4	1	6
	post	9	3	□-5=8	9	-	5+8=□	-	-	3

Table Preinstructional and Postinstructional Performances on Group Tests for Rerepresenting Level Students (N=11) - Change Problems

Verbal Problem Type		Correct Sentence	Correct Solution	Direct Representation			Rerepresentation			Other
				Form	Complete	Incomplete	Form	Complete	Incomplete	
Change 1	pre	7	6	8+7=□	7	-				4
	post	11	10	9+4=□	11	-				-
Change 2	pre	7	10		7	2				2
	post	10	7	12-9=□	10	1				-
Change 3	pre	2	6	9+□=12	--	5	12-9=□	2	-	4
	post	11	10	7+□=11	11	-	11-7=□	-	-	-
Change 4	pre	3	7	14-□=9	--	3	14-9=□	3	-	5
	post	11	7	15-□=8	10	-	15-8=□	1	-	-
Change 5	pre	1	4	□+6=13	--	5	13-6=□	1	-	5
	post	11	9	□+5=11	11	-	11-5=□	-	-	-
Change 6	pre	2	5	□-4=8	--	4	4+8=□	2	1	4
	post	11	5	□-5=8	11	-	5+8=□	-	-	-

THE FEASIBILITY OF ASSESSING 7-11 YEAR OLD PUPILS' UNDERSTANDING OF NUMBER IN A CLASS ADMINISTERED TEST

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Aim of Study and Rationale

The aim of this work was to:-

> **investigate the extent to which a class assessment instrument would yield reliable diagnostic information about individual pupil's mathematical understanding**

The potential value of a class assessment instrument is two-fold. It would provide: 1. a diagnostic assessment instrument for the teacher; 2. a data collection instrument for research into children's understanding of number.

Theoretical Stance

A constructivist view of learning, which has a Piagetian basis and is described by Schaeffer et al (1974) and more recently by von Glasersfeld (1985) is adopted in this study. Children are seen to construct their own knowledge by making links between previously unrelated ideas. The solutions which they offer will depend on their mental "re-presentation" (von Glasersfeld 1985) of the task, that is the overall definition or "meaning" which the task, as it is presented, evokes in the mind as a result of previous experience. The extent to which different individuals - pupils and teachers - attach the same meaning to a task, will depend on their common experience and the opportunity they have to negotiate meaning.

Research studies carried out for example by Carpenter and Moser (1982), Hart (1981) and Steffe (1983), based on individual interviews, have indicated that it is often possible to identify occasions when pupils are blindly following rules, situations where they have a clear grasp of "what they are doing and why" and numerous occasions when the response is along the continuum between these two extremes. Thus, whilst recognising that it is not possible or even meaningful to make a precise assessment of a child's understanding, it is hypothesised that for most children an individual interview will give a more accurate description than a class test.

In attempting to use assessment procedures in which meaning cannot be negotiated, there is likely to be an inaccurate description of the attainment of pupils whose perception of a task differs from that of the teacher. This study examines how closely pupils' performance in a class assessment of the understanding of number matches their performance in an individual interview when both are based on an empirically designed learning hierarchy.

Background and Methodology

This present study is based on the findings of earlier work (Denvir and Brown, 1986a). It is hypothesised here that some inferences about the strategies and perceptions which children have available for tackling questions about number may be made from their written responses to suitably designed items. Items were developed so that they could be administered to a whole class of 7-11 year old pupils. The items were orally and/or pictorially administered in the class test. A subsample were interviewed individually and performances in the two testing modes were compared.

The fourteen skills assessed are, in order of difficulty

1. knowledge of standard number word sequence
2. interpolation between decade numbers (whole number intervals)
3. estimation of numerosity of haphazard array of dots
4. cardinal representation of numeral
5. solution of addition and subtraction word problems
6. representation of multiplication word problems
7. enumeration of grouped collections
8. representation of addition and subtraction word problems
9. "ten more than" and "ten less than" two digit numbers
10. estimation of numerosity of cartesian array
11. mental subtraction of two digit numbers
12. representation of division word problems
13. interpolation between decade numbers (0.1 intervals)
14. solution of proportion problem

The details of the class assessment sample and the interviewed sub-sample are shown in Table 1. Performance in the interview was regarded as a reliable description of pupils' understanding against which performance in the class assessment could be compared.

Table 1

Number of Pupils in Main Sample by School and Year
Number of Pupils in Interviewed Sub-sample in brackets

School	Year of Junior School and Age Range 1 7.9-8.8	 2 8.9-9.8	 3 9.9-10.8	 4 10.9-11.8	Total
A	23*(7)	-	25(10)	-	48(17)
B	-	19(3)	19(3)	17(4)	55(10)
C	20*(6)	23(4)	22(5)	29(3)	94(18)
D	-	18*	17	17*	52
Total	43(13)	60(7)	83(18)	63(7)	249(45)

*In these classes four skills: 6, 12, 13, 14 were omitted from the assessment.

Figure 1 shows performance in the class test and the interview for the interviewed subsample who were assessed on every skill (n=32).

Figure 1

Skills Passed and Failed in Class and Interview by Age

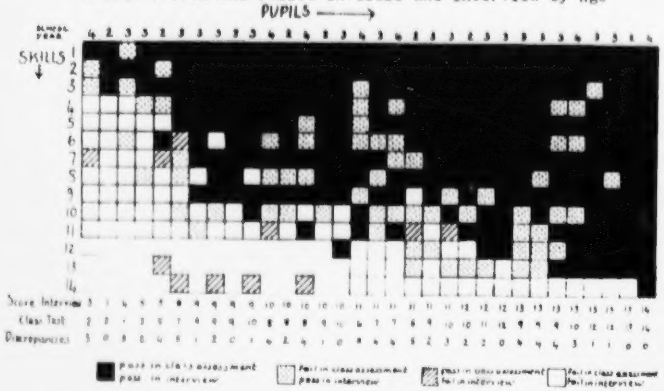
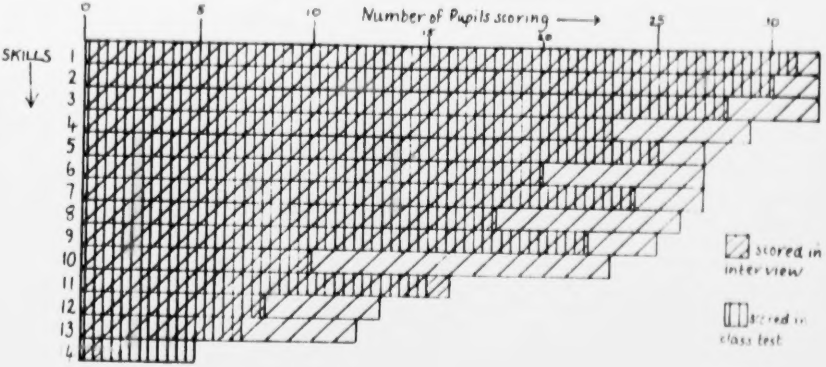


Figure 2 shows the number successful on each skill in the class assessment and the interview for the subsample assessment on all the skills.

For every child interviewed the interview responses which differed from the class result were examined in order to shed light on possible reasons for the variation and these are outlined in 'Discussion' below.

Figure 2

Number of Children Successful in Class Assessment and Interview



Performance on every skill except the hardest one was higher in the interview than the assessment. There was a total of 76 out of 448 (17%) pupil-skills on which performance differed in the class assessment and the interviews. Overall there was a reasonable correlation but for some individuals, especially those who performed badly on the class assessment, the discrepancy was unacceptably large. In particular there were six pupils (whose records of performance are asterisked in Figure 2) whose understanding would have been grossly misrepresented by the class test.

Discussion

The possible reasons for variations between response in the class assessment and the interview are:-

1. Learning. This may come about in the interview because the pupil tries to resolve inconsistencies.
2. Borderline knowledge. During construction of new knowledge, there is rarely an abrupt transition from "not knowing" to "knowing". More often the pupil appears to "know" on some occasions and "not know" on others.
3. Knowledge assessed in the class assessment is different from knowledge assessed in Interview. Asking the same question in a different context may stimulate quite different thought patterns in the child, which may produce different answers.
4. Anxiety greater in one context than another.
5. Failure to understand the point of the question in the class assessment.
6. Incorrect assumptions made about the necessary strategy for a correct solution.
7. Expectation of feedback in interview produces better motivation.
8. Inability to hear, see or attend in the class assessment
9. Accidental error in class.
10. Poor short term memory affects performance in the class.
11. Correctly guessing or estimating the answer in class.

Conclusions

1. It was not found possible to design written items for all of the number skills which the earlier study (Denvir and Brown, 1986a) had identified as crucial in the development of the understanding of number.

2. There were differences between pupils' performances in the class test and the interview. The number of differences varied according to the skill and the pupil.
3. Of the 32 pupils in the interview sample, six would have been significantly misplaced by taking performance in the class test rather than performance in the interview as a measure of their attainment in number. For each of these six pupils performance in the class test underestimated their understanding of number.
4. Nevertheless, the class assessment instrument designed in this study did provide an initial assessment of pupils' understanding of number from which pupils needing further diagnostic assessment could be identified. In particular, it was generally the case that if a pupil was successful on a skill in the class assessment then she was successful in the interview.

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HOW DO CHILDREN COPE WITH THE INFINITY OF NUMBERS?

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What does "understanding infinity" mean? We certainly do not grasp it by any sensory means nor by imagination, we *know* that there are infinitely many numbers, and that there does not exist a largest number. We conceive the infinity of numbers although we do not perceive it. Two sets of pioneering studies of children's ideas of infinity should be noted, one by Fischbein and his students and the other by Gelman and her associates. Fischbein et al. (1979) and Tirosh (1985) presented a variety of questions many of them concerning the unending divisibility of line segments (following the lead of Piaget and Inhelder, 1967) and similar geometrical problems. We chose to focus on potential infinity in the *numerical* area rather than on the infinite divisibility of line segments. A child may relate to the physical constraints of the task of dividing a segment, which is not what we are interested in, and rightfully claim that the partitioning will eventually terminate because "there is no more space". Moreover, we preferred not to rely on geometrical images in order to avoid the need to refer to idealized concepts like a point that has neither length nor area. Evans and Gelman (1982) interviewed young children about the "biggest number" and mapped several developmental stages, from "finite and small" through "finite and large" to "infinite", in children's understanding of infinity. We tried to devise a *choice situation* in which a child's understanding of infinity could be exhibited without having to answer direct questions about the largest (smallest) number. We reasoned that the comprehension of the idea that the integers are *unbounded* should be expressed not only by a readiness to recite the slogan that "there is no biggest number", but also by the ability to name a larger number than *any* number you may suggest. Our purpose was to find out whether children understand this principle so that they can profitably apply it in a competitive game, and whether they are able to verbalize it.

Method

A number-game played by the child and the experimenter was devised. The instructions ran as follows: "Each of us should say a number. The one whose number is larger will win. Would you like to be the first or the second?" After answering, and before playing, the child was asked to explain her choice. Upon choosing and playing several times, a "continuous game", in which both players were alternately naming increasing numbers for five or six rounds in succession, was played. At that point the child was asked for how long the game can go on and whether it will ever end. Next, a variation of the game in which the winner is the one whose number is the smallest was suggested. This game was played in several versions: one with only non-negative integers, another with integers including negative numbers, and a third version with only positive numbers including fractions. The versions played with each child were determined by the child's ability to manipulate negative numbers and/or fractions. The dialog was not strictly structured. We tried to follow the spontaneous and free expressions of the children, sometimes at the expense of missing responses to some of our intended questions.

Results

The preliminary results of 95 children, 56 girls and 39 boys, aged 5(0) to 12(6), were analysed. The three games involving a possible endless succession of numbers were labeled:
1. Upwards - Natural Numbers
2. Downwards - Integers
3. Downwards - Positive Rationals.

All children but one played game 1; 33 played game 2, and 48 game 3. Every child was assigned three scores for three functions involved in each game: performance in the game, ability to verbalize the principle of winning, and understanding the boundlessness of the numbers and/or the idea that the game may go on indefinitely (the possible scores were +, -, or ?). The distribution of the percentages of positive scores by grade (age) and task (i.e., game and function) is presented in Table 1. The percentages were computed out of the available responses in that grade and task. Empty cells represent cases where less than a third of the children in the grade responded to the task.

A general developmental trend is apparent. Preschoolers and some children in grades 1-3 could hardly play even the continuous part of game 1. Most 6 and 7 year olds failed in understanding the boundlessness of the integers. This understanding, and the ability to explain why one should play second, rise steadily with age and characterize the majority of the 11-12 age group. The conception of infinity in games 2 and 3 appears later than in game 1, partly because children become familiar with negative numbers and fractions only in older ages. However, the results suggest that once they learn about negative numbers they easily perceive their symmetry to positive numbers all the way to infinity. Understanding of the infinitesimal appears to pose additional difficulties. Quite a few subjects could understand the endlessness of the process in game 1 but not in game 3; there were no cases where the reverse was true. By and large, correct choices in a game preceded formulations of the principle of winning, which in turn preceded the understanding of the infinity of the process.

Table 1. Percentage of positive scores (out of the available responses in a given category), by grade (age) and task.

Grade Age range	Type of game and function								
	1. Upwards			Downwards					
	Natural Numbers			2. Integers			3. Positive Rationals		
Number of children	Performance	Principle	Infinity	Performance	Principle	Infinity	Performance	Principle	Infinity
Kindergarten 5(0)-6(0) n=4	25	0	0						
1 6(1)-7(6) n=13	69	60	31						
2 7(0)-8(6) n=13	77	70	46						
3 8(0)-9(2) n=11	64	71	55				75*		
4 9(3)-10(6) n=11	91	70	73	100*	100*	100*	50*		50*
5 9(5)-11(2) n=21	100	93	81				100	100*	72
6-7 10(11)-12(6) n=22	100	92	86	100	100	78	100		61

* Out of less than half the subjects of that grade.

Verbal responses. Many responses were informative and instructive. Consider a few examples (the games to which the reponses refer are designated by their numbers):

Negative scores: Shoshi's 8(6) answer to the question why she chose to be first in the continuous part of game 1, was: "Because I thought that you'll be the first to run out of numbers." Shoshi seemed on the alert to seize the "last number", however, she was not successful. When we got to 10,000 she said "I have no more numbers ... I don't know whether there are numbers beyond 10,000." Ron 11(0), 2.

"... just as we'll eventually reach the end of big numbers, so we'll get to the end of negative numbers. It is the same, only one had put a minus in front of it." Erez 12(5), 3. "If we play a long time the numbers will end because they get smaller and smaller."

Positive scores: Itai 6(6), when asked how long game 1 may go on: "Until no end, the numbers will never stop, because if there is, say, a milliard, one can add to it another and another milliard, until infinity." When asked what happens if we add one to infinity: "Nothing, because infinity is not a number." 3. Itai wanted to be second. He responded to 1 by ½ and to ¼ by "a quarter of a half ... I'll take a smaller and smaller grain, each time still smaller ...". Avi 7(7), 1. "We'll not finish playing, because if you say, for example, a number, I'll say a greater one, and then you'll have a greater one, and then I'll say still a greater one." Yechi 8(6), 1. "It is better for me to be second, because you'll reveal your number and so I'll win, but if I am the first I reveal my number, and you win." Sigal 10(6), 1. "I want to play second because I'll know what you said first, and I'll be able to plan which greater number to say." 2. "I think one cannot finish this game either, because you simply take all the numbers, as before, and transfer them backwards behind zero." Yoram 11(11), 3. "I think we'll never reach zero, we'll only get closer to it each time ... even though the interval looks small it includes all the numbers in the world, because each number you can turn into one divided by the number." Oren 12(0), 2. "Big deal! I'll always be second and so I'll say a smaller number than yours, because to each big number you can add a minus and it becomes small."

Unexpected responses: Four children, aged 5(6) to 8(6), offered fractions of zero in response to zero in the downwards game. Haim 7(1): "Half a zero ... ½ of a zero is smaller than zero because it is not a whole zero but only part of it." These children were ready with the algorithm "whatever you say I'll respond by half of it", it was only the concept of zero that was not clear to them. Emmanuel 11(11) wanted to be second and knew why, however, he refused to play the games, because "it's boring". Emmanuel had a suggestion of his own: both players will write their numbers on notes and then we'll see which number is higher. This would be "fair". We followed his advice and let a few subjects play also with notes. Then we asked them which kind of game is "fairer". They thought that the notes game is fairer, like Shimrit 7(6): "In the previous game the first one always loses, and here each one writes and doesn't know what the other one is writing, and you can lose and you can win."

Concluding comments

It was surprising to realize the importance that several children attached to knowing the name of the numbers, as if the concept exists via its name. In contrast to Irene 9(8) who said "if there is a number, then there is also a greater number", Roi 8(4), said: "I don't know a number greater than a milliard, I don't know how you name them ... we don't know the largest number. There is a certain number but its name is unknown, nobody knows." Sarit 10(2) solved that difficulty elegantly: "you can never finish such a game, you can always add more and more numbers and invent more names for them as long as you wish."

Children's attempts to cope with the conflict between the finiteness of everything around and the awareness of the possibility of the infinite, were often manifested by contradictory statements within an answer, or by gaps and inconsistencies between answers. The intermediate stage between mastering finite number schemas and the breakthrough to infinity is of special interest. One has to make a discontinuous leap between these two kinds of concepts, since nowhere does the very big start to merge into the infinite. In order to find out when that unbridgeable gap becomes clear to children, we started presenting paired comparisons between finite and very large sets and the "smallest" possible infinite set. Noga 5(9) first decided that there are more grains of sand in the world than leaves on all the trees, than hair on all people's heads, and than words in all the languages. But when asked "What are there more - grains of sand, or numbers?" she chose numbers decisively "because the numbers never end ... but the grains of sand always stay as they are." This bold statement did not stop Noga from subsequently choosing to be first in our games both upwards and downwards.

Although the spatial-geometrical context had been neglected as a formal medium of inquiry in our research, a few children brought up geometrical images in the course of the interviews. Rinat 11(2), 1. "One cannot finish playing, because it is like a building where you can pile more and more bricks on top of each other and it never stops." Smadar 9(10), 3. "I think that if we'll go on we could finish the game, because there is little space between 0 and 1 and there are not so many numbers there." Yifaat 10(11) also thought one could finish counting all the numbers between 0 and 1 "because an interval is many points combined together, and if each point is a number we could count all the points and this way we'll count how many numbers there are in the interval." Apparently, geometrical images like "space" and "point" interfere with the conception of the infinitesimal that should be free of all physical constraints.

Paradoxically, understanding infinity, like other Piagetian concepts, can be analysed in terms of *conservation*. One has to realize that infinity is unchanged by addition/subtraction of a finite (and even infinite) number. This is the heart of the problem and the most unintuitive characteristic of an infinite set. Some of our subjects were apparently aware of the "conservation of infinity" when they objected to our suggestion to add any number to "infinity" in an attempt to get a bigger number. Nir's 9(1) response to such a suggestion was "that is impossible! infinity is something greater than all numbers." When asked whether infinity is a number, he said "no, infinity means, for example, that you get to a million and you go on counting without ever stopping."

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First and Second Graders' Performance on Compare and Equalize Word Problems

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First and second-graders' performance on Equalize was superior to that on Compare problems, due largely to children's inability to understand the "more than" or "fewer than" relationships expressed in Compare problems. Performance as a function of missing story element depended on the child's overall preference for using addition; young children especially tended to perform well only where addition was required. Generally, children performed better on problems in which the necessary solution strategy agreed with the directionality ("more" or "less") implied in the given problem.

This report describes studies in which performance of first and second graders on Compare and Equalize word problems was examined. Difficulties with solving Compare and Equalize problems were a particular focus of the present study because these types are not well represented in American textbooks, while Compare problems are very well represented in textbooks in the Soviet Union (Stigler, Fuson, Ham, & Kim, 1986). Furthermore, many of these forms have not been well studied to date, especially the "less" forms and the Equalize problems.

Problem Subtypes

Written tests of word problems that used sums and differences of 10 and less and that crossed the three missing numbers (Missing Difference, Missing Small, and Missing Large number) with problem type (Compare or Equalize) and with direction of the question form (More Than or Less/Fewer Than) were given early in the year to seven first- and second-grade classes in order to obtain basic information on performance (see Table 1). Two basic dependent variables were analyzed: correctness of strategy (answer within 3 of the correct answer, which served to differentiate adding from subtracting) and correctness of answer.

A significant superiority of Equalize over Compare was found for strategy in 5 of the 7 classes tested (classes 2HZ, 2AZ, 2AY, 1AZ, and 1AY), all F's>5.04, $p<.05$. Equalize was also superior to Compare for the answer measure in 5 classes, all F's>4.34, $p<.05$ (classes 2HZ, 2AZ, 2LY, 1AZ, 1AY). Thus, a problem which states a comparison in terms of the action required to remove the inequality was easier for children than one which merely expresses the comparison relationship.

When all classes were combined, the effect of missing position was significant, reflecting better performance on the Missing Big forms for both strategy and answer. However, analyses by individual class showed this to be true only for the first graders and the lowest level second graders. Examination of

Table 1

Different Forms of Compare and Equalize Word Problems

Compare	Equalize
Missing Difference (More) Jane has 7 dolls. Ann has 3 dolls. How many more dolls does Jane have than Ann?	**Missing Difference (More)** Jane has 7 dolls. Ann has 3 dolls. How many dolls does Ann have to get to have as many dolls as Jane?
Missing Difference (Fewer) Jane has 7 dolls. Ann has 3 dolls. How many fewer dolls does Ann have than Jane?	**Missing Difference (Fewer)** Jane has 7 dolls. Ann has 3 dolls. How many dolls does Jane have to lose to have as many dolls as Ann?
Missing Small (More) Jane has 7 dolls. Jane has 4 more dolls than Ann. How many dolls does Ann have?	**Missing Small (More)** Jane has 7 dolls. If Ann gets 4 more dolls, she will have as many dolls as Jane. How many dolls does Ann have?
Missing Small (Fewer) Jane has 7 dolls. Ann has 4 fewer dolls than Jane. How many dolls does Ann have?	**Missing Small (Fewer)** Jane has 7 dolls. If Jane loses 4 dolls, she will have as many dolls as Ann. How many dolls does Ann have?
Missing Big (More) Ann has 3 dolls. Jane has 4 more dolls than Ann. How many dolls does Jane have?	**Missing Big (More)** Ann has 3 dolls. If Ann gets 4 more dolls, she will have as many dolls as Jane. How many dolls does Jane have?
Missing Big (Fewer) Ann has 3 dolls. Ann has 4 fewer dolls than Jane. How many dolls does Jane have?	**Missing Big (Fewer)** Ann has 3 dolls. If Jane loses 4 dolls, she will have as many dolls as Ann. How many dolls does Jane have?

Compare and Equalize problems all involve the comparison of two amounts (which we have termed Big and Small) and the difference between these amounts (Big - Small = Difference and Small + Difference = Big).

Table 2

Percent of Correct Strategies and Correct Answers on Compare and Equalize Problems by Class

Grade	Ability[d]	School	Compare						Equalize					
			Missing D[a]		Missing S[b]		Missing B[c]		Missing D[a]		Missing S[b]		Missing B[c]	
			More	Fewer	More	Fewer	More	Fewer	More	Fewer	More	Fewer	More	Fewer
			Correct Strategy											
2	H	Y	100	93	81	93	93	52	100	100	70	96	100	96
2	A	Y	85	100	38	46	85	62	85	92	62	77	100	77
2	A	Z	74	68	47	63	74	37	63	74	68	63	74	74
2	L	Y	25	100	0	38	50	50	75	63	50	75	75	13
2	L	Z	23	31	8	8	83	85	8	15	23	23	77	85
1	A	Y	20	47	7	7	53	33	67	60	20	33	73	47
1	A	Z	32	32	27	9	55	50	59	36	27	27	64	77
			Correct Answer											
2	H	Z	100	93	74	93	93	48	96	100	67	96	100	87
2	A	Y	85	100	38	46	85	46	85	92	46	62	100	77
2	A	Z	68	58	37	53	68	16	58	74	53	47	58	63
2	L	Y	25	100	0	38	38	25	63	63	38	75	75	13
2	L	Z	15	31	8	0	62	54	8	15	8	23	39	62
1	A	Y	20	20	0	0	27	13	53	53	5	13	67	27
1	A	Z	18	14	14	5	36	23	23	32	67	14	59	41

a. Missing Difference b. Missing Small c. Missing Big d. H: High, A: Average, L: Low individual student papers revealed that many children in these classes added on all problems, resulting in higher performance on the one missing position form solved by adding: the Missing Big problem. In general, as age and ability level increased, there was a tendency for the Missing Difference form to become easier than the other forms. Thus, as the children got more experience with subtraction, they improved more on the Missing Difference form than on the others. No significant main effect of varying the direction of the wording in the problem (More or Fewer) was found, either overall or for any of the 7 classes analyzed.

Performance as a function of specific problem subtype varied markedly according to class. The independent variables did interact with one another, but often did so in an unstable manner; both the presence and direction of these interactions tended to depend on class in a manner not easily explained. This suggests that these various effects are either unstable or interact further with ability level and prior learning. An exception was the Missing-Position X More-Fewer interaction, which was significant for strategy and answer in 5 of the 7 classes (2HZ, 2AZ, 2AY, 2LY, 1AY). In all 5 classes, performance on the forms involving the "more" direction increased linearly from S to D to B, while performance on the "fewer" forms was higher for the Missing Difference form than for the others. Overall, the Missing Big "more" forms produced considerably higher performance than the Missing Big "fewer" forms, and the Missing Small "fewer" forms were somewhat easier than the Missing Small "more" forms. Thus, performance was better where the directionality of the problem parallelled the strategy necessary for solution ("more" implies adding and "fewer" subtracting). Interestingly, for classes showing this interaction, performance on the Missing Difference problems was better for the "fewer" than for the "more" forms wherever there was no ceiling effect. Because this form requires subtraction for the solution strategy, it appears that the "fewer" structure may have acted as a cue to subtract.

Interview Studies

Interviews were held with 23 first graders who had solved correctly at least one problem on a written test. The four simplest (Missing Difference) forms of the Compare and Equalize problems were given. Children were first required to retell the given story (e.g., see De Corte and Verschaffel, 1985a) and then to represent and solve the problem using physical models of people and objects that represented people and objects in the problems. It was found that more correct retellings, strategies, and answers were given for the two Equalize stories than for the two Compare stories: 27 and 10, respectively, for the retellings, $F(1, 22) = 16.82$,

$p<.001$; 34 and 19 for strategies, $F(1, 22) = 12.50$, $p<.002$, and 30 and 15 for answers, $F(1, 22) = 14.14$, $p<.001$. Analysis of the retellings and modelled solutions indicated that many beginning first graders could not solve these Compare stories correctly because they lacked an understanding of the meaning of the questions "How many more (less) does Susan have than Jane?" Children understood the Equalize forms of these questions ("How many does Jane need to get (lose) to have as many as Susan?") better. By far the most frequent error made for these problems was to answer with one of the given numbers (this occurred 17 times) as if the child was ignoring the comparison and was just answering the question "How many does Susan have?"

Interviews were also collected from 31 second graders on the more difficult Missing Big and Missing Small forms. Interviewed children were those who showed discrepant scores on Equalize and Compare problems or who got all problems correct on a written test. The results indicated that although children were more accurate in retelling the Compare (19 correct) than the Equalize (10 correct) forms of these problems, $F(1, 23) = 4.97$, $p<.04$, strategies were correct marginally more often for Equalize (27) forms than for Compare (20) forms, $F(1, 23) = 3.62$, $p<.07$, and answers were nonsignificantly higher for Equalize forms (24 versus 19). The superiority in retelling of the Compare story seemed to occur mostly because of the length and verbal complexity of the Equalize forms. Many incorrectly retold Equalize stories were simply incomplete; the retellings often reflected the total structure of the Equalize problem (the child retold the first two sentences) but omitted the last question. The most common retelling error for both Compare and Equalize stories was to give each person in the story one of the numbers in the story, i.e., to reduce the story to the simple Missing Difference form. This indicates that, when presented with a phrase such as "Mary has 5 more than Jane", children often have difficulty in simultaneously comprehending both the inequality (that Mary has more than Jane) and the amount of the inequality (5 more), and so simply encode and/or remember the quantity as belonging to the person mentioned ("Mary has 5.").

Conclusions

The results of the present studies point to several conclusions related to story problems in general and to Compare and Equalize types specifically. First, it appears that the general assumption that semantic types of word problems are stable in their degree of difficulty is a false one. Performance as a function of problem semantic variables may vary markedly with such factors as ability level, teacher, math topics just learned at the time of testing (especially addition and subtraction), and topics already learned; time of year in school may influence the last two variables considerably. Especially important is the finding that the inherent "difficulty" of various problem types depends strongly on the child's prior knowledge of addition versus subtraction and thus on his or her tendency to favor one operation or the other. Thus, a word problem may be difficult not because of semantic components inherent in that problem but because the required solution strategy is inconsistent with the one preferred by the child at that time. However, such preferences seemed to be held much more by young or by slow learners. Older and brighter children did not routinely use one operation almost to the exclusion of another. Second, although Compare and Equalize word problems may be structurally similar, they are not responded to in an identical manner by children. First graders have little trouble comprehending Equalize stories, but many do not understand the "more than" relationship expressed in Compare stories. Even some second graders show difficulty in comprehending the "more than" relationship expressed in the difficult Compare stories. Thus, the most consistent effect in these studies was one of a superiority of performance on Equalize over that on Compare problems. Third, as has been reported previously (De Corte & Verschaffel, 1985b; Willis & Fuson, 1985), problems in which the solution strategy conflicts with the implied directionality of the problem action are difficult for children.

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CHOICE OF OPERATIONS: FROM 10-YEAR-OLDS TO STUDENT TEACHERS

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Fischbein, Deri, Nello and Marino (1985) have proposed a theory to account for observed results when children are presented with single-operation verbal problems involving multiplication and division and asked to specify which operation would produce the answer. In particular, they hypothesized that the primitive model associated with multiplication is repeated addition, in which a number of collections of the same size are put together. In this situation, the number of objects in each collection (the **multiplicand**) clearly plays a different role from that of the number of collections (the **multiplier**). A consequence of this is the distinction between partitive and quotitive division when a collection is split up into equal subcollections. Partition and quotition are hypothesised as the two primitive models for division.

The theory postulates that choice of operation will prove easy if the situation can be assimilated to the appropriate primitive model. The consequent numerical constraints are as follows:

(a) multiplication: the multiplier must be an integer (though for older pupils it is suggested that an "absorption effect" may operate for decimals where the whole part is clearly larger than the decimal part).

(b) partitive division: the divisor must be an integer. (Fischbein et al also propose that the dividend must be larger than the divisor, but this is by no means clear, and will not be assumed here).

(c) quotitive division: the dividend must be larger than the divisor.

Greer (in press) suggests that the multiplicand/multiplier distinction may be extended to more complex multiplicative structures such as Change of Scale, Rate and Measure Conversion. For example, if a man walks for 3 hours at 4 miles per hour, then the 3 plays the role of the multiplier and the 4 that of the multiplicand. Moreover, the definitions of partition and quotition can be generalized by defining the former as division by the multiplier and the latter as division by the multiplicand.

The experiment to be reported here extended the empirical investigations reported by Fischbein _et al_ in several respects:

1. A wider developmental range was tested.
2. A more varied and systematic sampling of number combinations was used.
3. More complex contexts were used, namely Measure Conversion (specifically currency conversion), Price (unit price/quantity/total cost) and Speed.

METHOD

Paper-and-pencil tests were administered to six groups: 10-year-olds (n = 68), 2nd form pupils (n = 82), 4th form pupils (n = 73), pupils at a College of Further Education (n = 54), first year psychology students (n = 50) and students training to be primary school teachers (n = 50). Three tests with 40 items in each were given on separate occasions. Each item was a verbal problem involving two numbers for which the subjects had to specify the operation which would yield the correct answer, but were not required to calculate the answer. The 120 items included 8 multiplication items, 8 partitive division items, and 8 quotitive division items for each of the three contexts. Addition and subtraction items were also included so that it could not be assumed that the correct operation was always multiplication or division. Classes of comparable number combinations were constructed so that incidental features of particular combinations of numbers would not bias the results. (For details of the matching of items, counterbalancing of order, definitions of number classes etc., see Mangan (1986)).

RESULTS

Multiplication: For each of the 6 groups, 3 contexts and 8 number classes the facility (percentage of correct answers) was calculated, yielding 144 facilities in all. Graphical analysis made it clear that the pattern of results across number types was remarkably consistent regardless of the subject group and context being analyzed. A regression analysis was therefore carried out, with the 144 facilities as the criterion variable, and groups, contexts, type of multiplicand, and type of multiplier, as predictors. This confirmed that an additive model provided a very good fit (accounting for 92% of the variance). The results can therefore be summarized in terms of the overall means for each level of each factor, averaging over all the other factors. The first column in Table 1 shows that there was the sort of developmental improvement in performance that could have been expected, and that the currency conversion problems proved rather more difficult than the speed and price problems.

Table 2 shows a clear difference in the size of effects associated with the multiplicand and multiplier. The type of number used as multiplicand had virtually no effect, whereas changing the multiplier from an integer to a decimal less than 1 lowered performance by about 46% across the board.

Division: Additive models were found to fit the results for division equally well (for separate regression analyses for partition and quotition, in each case the proportion of variance accounted for was 92%). The rest of Table 1 shows similar developmental and contextual effects as for multiplication, with only small differences between the results for partition and quotition. The first two columns of Table 3 show the facilities for partition and quotition for the 8 number classes used. Again the differences between partition and quotition are rather small. A further analysis was carried out on the types of error made. Most of the errors consisted of either choosing multiplication or of reversing the numbers to be divided. For each number class, the number of reversal errors was calculated as a percentage of the number of reversal and multiplication errors combined; the results are shown in the second part of Table 3.

DISCUSSION

The results for multiplication vindicate the extension of the multiplicand/multiplier distinction and are entirely consistent with Fischbein's theory since they clearly demonstrate the crucial importance of the constraint that the multiplier be an integer.

The results for division are less clearcut, since the differences in facility between partition and quotition items involving the same number classes were rather small. However, it is noticeable that there are three types for which performance for quotition items was better by about 10% overall. These include two types which satisfy the constraint for quotition, but not that for partition, namely 32/5.69 and 8/0.77; moreover, a similar explanation may be advanced for the third such type, 5.87/0.44, by invoking the absorption effect.

Thus, there is some evidence that the subjects were sensitive to the structural difference between partitive and quotitive items, although it seems likely that many of the errors were due to responding superficially to the two numbers stated in the problem, and the relationship between them. (Although Fischbein *et al* play down the role of numerical misconceptions, there is clear evidence that they are implicated in many errors).

Further evidence that the partitive/quotitive distinction was affecting performance was provided by the analysis of the types of error made (Table 3). The two types 32/5.69 and 8/0.77 are difficult to assimilate to the partitive model, hence there is a strong tendency to "force" them into this model by reversing the order of the numbers. The results confirm that most of the errors in these cases were indeed reversals. By contrast, both types *can* be assimilated to the quotitive model, and in this case only a small proportion of errors were reversals.

The remarkable consistency of the patterns of results across number types for all age groups provides strong support for Fischbein's belief that the primitive operations continue to affect the interpretation of multiplicative situations even after extensive formal training. Fischbein *et al* suggest that what is needed is an attempt to "provide learners with efficient mental strategies that would enable them to control the impact of these primitive models" (p.16). Arguably, an alternative strategy would be to introduce a wider range of multiplicative structures as early as possible. Particular attention also needs to be given to the critical point at which the concepts of multiplication and division are generalized from the domain of integers to that of the rationals.

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Table 1. Overall means for groups and contexts

	Multiplication	Partition	Quotition
Student teachers	79	72	77
Psychology students	76	61	66
P.E. students	68	55	58
4th form pupils	65	54	59
2nd form pupils	57	43	45
Primary pupils	41	25	25
Price	71	61	63
Speed	67	48	46
Currency conversion	56	46	56

Table 2. Overall means for different types of numbers

	Integer	Decimal > 1	Decimal < 1
Multiplicand	67	64	63
Multiplier	84	71	38

Table 3. Results for division items

	Facilities		100R / (R + M)	
Number class	Partition	Quotition	Partition	Quotition
25/8	79	80	31	18
26.85/9	85	84	32	21
11.44/4.51	74	75	(not enough for stable estimates)	
32/5.69	50	61	85	31
5.87/0.44	54	62	35	10
8/0.77	31	41	67	17
7/23	22	18	92	78
0.39/0.89	19	20	68	67

(R = number of reversal errors, M = number of multiplication errors)

THE DEVELOPMENT OF NUMERATION CONCEPTS: SCHOOLCHILDREN AND THEIR CURRICULUM

Bruce Harrison, Marshall P. Bye, and Thomas L. Schroeder

ABSTRACT

This paper reports percentage distributions by cognitive level of the responses that 500 six- to eleven-year-olds made to Numeration tasks that are central to the elementary school curriculum. The levels of pupil response are then contrasted with the cognitive demand levels of the relevant curriculum objectives, textbook material, teacher presentations, and achievement test items.

Is there a reasonable fit between the *instructional cognitive demands* in Numeration topics for six- to eleven-year-olds and the levels of *pupil cognitive response*? Pursuing one of the PME goals, to better understand the psychological impacts (and implications) of learning and teaching mathematics, the preceding question was addressed in a two-year cognitive assessment project (Marchand, Bye, Harrison, & Schroeder, 1985) which was designed on principles drawn from constructivist theories about how children learn mathematics (e.g., those of Piaget and Skemp).

COGNITIVE ASSESSMENT

Cognitive assessment procedures were developed for principal Numeration topics in each of the six elementary school grades. The assessment interviews were drawn largely from the work of Jean Piaget and from interviews reported by Robert Davis. In all, 594 Numeration interview task assessments were made of the responses from 360 Grade 1 to 3 pupils. The interviews were conducted by 23 Teacher-Interviewers who had participated in four day-long workshops led by the investigators.

Most of the items for the paper-and-pencil assessment tests were selected or adapted from the Australian Council for Educational Research (ACER) *Mathematics Profile Series* based on Collis' research findings, with additional items adapted from the Chelsea Diagnostic Mathematics Tests (Hart et al., 1985). The Grade 3 to

6 cognitive response levels of 391 pupils were assessed by means of a Grade 3/4 or 5/6 Numeration paper-and-pencil test.

The criteria imbedded in the pupil response assessments were also used to assess the cognitive demands made by the Numeration curriculum objectives, textbook materials, teacher presentations, and Alberta Education Achievement Tests.

Sample interview tasks, test items, and cognitive demand criteria are given in Harrison, Bye, & Schroeder (1985, p. 211-213) and in Marchand, Bye, Harrison, & Schroeder (1985).

RESEARCH QUESTIONS

The information from the assessments was analyzed to provide answers to the following questions:

1) What levels of cognitive ability are demonstrated in Numeration topics by Alberta pupils in each of Grades 1 through 6 (Ages 6 through 11)?

2) What are the levels of cognitive demand made on pupils in Numeration topics at each grade level by:
   - i) the objectives identified by the Elementary Mathematics Curriculum Guide, Alberta Education, 1982,
   - ii) the prescribed textbooks,
   - iii) teacher presentations, and
   - iv) achievement tests?

3) How well do the curricular demands (made by the curriculum objectives, texts, teacher presentations, and tests) fit the distributions of pupil cognitive responses in Numeration topics at each grade level?

COGNITIVE RESPONSE AND DEMAND FINDINGS: NUMERATION

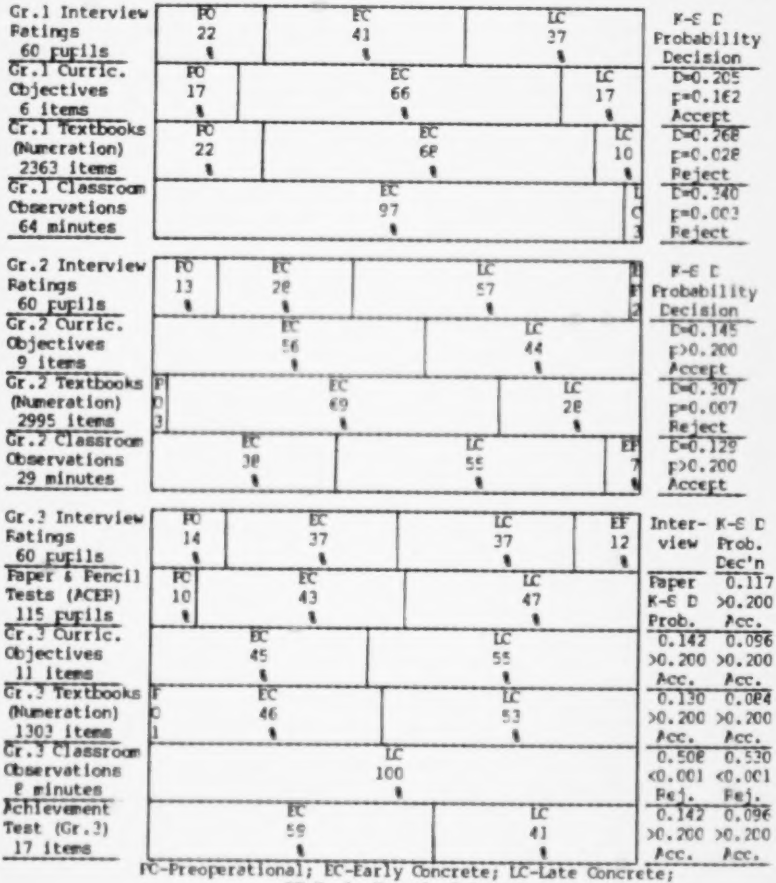
Summaries of the findings relevant to Questions 1) to 3) are presented in Figures 1 and 2 on the following pages.

Regarding Question 1), it was found that from 74 to 90% of the *pupil responses* were at the Concrete Operational level. The remaining responses were primarily at the Preoperational level in the early grades (10% to 22%) and at the Early Formal Operational level in the higher grades (3% to 13%).

Tabulation of the data relevant to Question 2) showed that 9 to 100% of the

Figure 1

Curricular Demand and Pupil Response Contrasts: Numeration, Grades 1 to 3



PO-Preoperational; EC-Early Concrete; LC-Late Concrete;
EF-Early Formal; F-Formal

Figure 2

Curricular Demand and Pupil Response Contrasts: Numeration, Grades 4 to 6

Gr.4 P&P Test Ratings 94 pupils	PC 1 %	EC 28 %	LC 59 %	EF 13 %		K-S D Probability Decision
Gr.4 Curric. Objectives 11 items		EC 18 %	LC 55 %	EF 27 %		D=0.145 p>0.200 Accept
Gr.4 Textbooks (Numeration) 2321 items		EC 22 %	LC 55 %	EF 23 %		D=0.098 p>0.200 Accept
Gr.4 Classroom Observations 72 minutes		EC 10 %	LC 47 %	EF 43 %		D=0.303 p<0.001 Reject

Gr.5 P&P Test Ratings 92 pupils	PC 16 %	EC 42 %	LC 39 %	EF 3 %	K-S D Probability Decision
Gr.5 Curric. Objectives 13 items		EC 15 %	LC 31 %	EF 54 %	D=0.506 p<0.001 Reject
Gr.5 Textbooks (Numeration) 1662 items		EC 9 %	LC 52 %	EF 39 %	D=0.485 p<0.001 Reject
Gr.5 Classroom Observations 86 minutes			LC 38 %	EF 62 %	D=0.584 p<0.001 Reject

Gr.6 P&P Test Ratings 90 pupils	PC 14 %	EC 17 %	LC 57 %	EF 12 %	K-S D Probability Decision
Gr.6 Curric. Objectives 14 items		EC 14 %	LC 36 %	EF 50 %	D=0.378 p<0.001 Reject
Gr.6 Textbooks (Numeration) 2591 items		EC 1 %	LC 15 %	EF 84 %	D=0.718 p<0.001 Reject
Gr.6 Classroom Observations 135 minutes			LC 9 %	EF 91 %	D=0.789 p<0.001 Reject
Achievement Test (Gr.6) 11 items		EC 9 %	LC 27 %	EF 64 %	D=0.514 p<0.001 Reject

PC-Preoperational; EC-Early Concrete; LC-Late Concrete; EF-Early Formal; F-Formal

Numeration *demands* were at the Concrete Operational level. Of the remaining demands, up to 91% were Early Formal at one grade level, with small percentages of Preoperational and Formal Operational demands. The distributions of *curriculum objective* demands showed the most consistent pattern, becoming increasingly demanding with increasing grade level. The *textbook* and *classroom* demands were predominantly Concrete Operational up to Grade 4 and predominantly Early Formal in Grades 5 & 6 (excepting Grade 5 *textbooks*). The Grade 3 & 6 *achievement test* demand distributions were generally comparable to those of the *textbooks*.

As for Question 3), two-sample Kolmogorov-Smirnov tests were used to determine whether or not there were significant differences between the distributions of pupil responses and those of the relevant curricular demands. Two distributions were considered to be not significantly different if the probability of observing the calculated K-S D was greater than 0.05.

Mismatches attributable to disproportionately high percentages of Early Formal Operational demands were found in the Numeration *curriculum objective* demands, as compared with the responses of the Grade 5 & 6 pupils (Ages 10 & 11). In Grades 3 & 4 (Ages 8 & 9), there were reasonable matches between *textbook* demand and *pupil* response distributions, but in the other four grades there were significant mismatches attributable to too few Late Concrete Operational demands for the younger children and too many Early Formal Operational demands for the older pupils. Except in Grade 2 (Age 7), the *classroom* demands were invariably pitched at cognitive levels well beyond the response levels of most pupils. The *achievement-test* demand distribution matched the *pupil* response distribution reasonably well at the Grade 3 (Age 8) level, but at Grade 6 (Age 11) there were too many demands at the Early Formal Operational level.

RECOMMENDATIONS

The findings support a recommendation that the Numeration *curriculum* *objectives* be reviewed and revised where necessary to provide more adequately for the cognitively less able, particularly in Grades 5 and 6, and for the cognitively more able, particularly in Grades 1 and 2. Also supported is a recommendation that the *textbooks* be supplemented to provide more learning experiences at the lower demand levels, especially the Preoperational and Early Concrete Operational levels. Furthermore, it is recommended that learning activities promoting the development of cognitive structures arising from Numeration topics be provided for every *classroom*. The kinds of materials envisaged are best exemplified by those produced by Richard Skemp in the Primary Mathematics Project.

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CHILDREN'S UNDERSTANDING OF RATIONAL NUMBERS - AN EMPIRICAL INVESTIGATION

Gisela Heink

A report on an empirical investigation of a current project, run by scientists of different fields of the Freie Universität Berlin and of the Technische Universität Berlin concerning difficulties in solving mathematical problems.
G. Ebneth; G. Heink; U. Lehnert; G. Leßner; R. Spann; R. Spyra; W. Reitberger

The problem - questions and hypotheses

It has often been observed that pupils were having enormous difficulties in understanding the rational number concept. Empirical investigations by K.M. Hart (3) and (4), K. Hasemann (5, 5a) and Th.R. Post, J. Wachsmuth, R. Lesh and M.J. Behr (6) and (7) - based on analysis of specially elaborated tests, on interviews with pupils and on direct observation - have shown considerable discrepancies between the concept of rational number the teachers believed to have tought and the actual concept the pupils had acquired.

The object of our investigation is to conceive these discrepancies in a systematical way. Our intention was to find out whether the aspects the teachers concider as contributing mainly to structuring the concept of rational numbers will be the same the pupils recognize as essential, and - if they are identical - whether they are weighted in the same way by teachers and pupils.

Issues of the excellently documented (8),(9) international reports on the last few years research led us to choose "fraction", "quotient", "ratio" and "chance" as basic concepts.

At school, these concepts are tought in connection with certain realizations: kilogramme, kilometre,..., geometrical representations, and texts. The geometrical representations are: circle, rectangle, straight line and line segment. Comparison and equivalence are essential topics. Calculation was excluded because there are many reports on research in addition, multiplication, etc., We are not so much interested in difficult calculation with rational numbers, but in the understanding of the concept of rational numbers.

The result of a structural analysis of our own concept of rational numbers is represented by the following scheme of categories:

fig. 1

type of item A	$? \xrightarrow{\text{operator}}$ part of the whole A_1 whole ?	whole $\xrightarrow{?}$ part of the whole A_2 operator ?	whole $\xrightarrow{\text{operator}} ?$ A_3 part of the whole ?
representation of the rational numbers B	geometrical representation B_1	quantities B_2	numbers B_3
question for the whole E	direct question or the whole is given E_1	indirect question it is neccessary to evaluate the whole for solving the task E_2	
representation of the item T	text essential T_1	text not essential T_2	
basic concept V	part - part = ratio V_1	part - whole = fraction V_2	quotient V_3 chance = probability V_4
number of steps necessary for solving the item S	one step S_1	more than one step S_2	more than one step and comparison S_3

B_1 can be devided into B_{1a} : circle
B_{1b} : rectangle
B_{1c} : line segment
B_{1d} : straight line

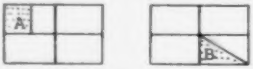
This scheme of categories represents our hypotheses. The intention of our empirical investigation is to find out whether this scheme of categories corresponds to the pupils' concept of rational numbers.

Outline of the investigation

The test:

We first constructed a test consisting of 60 items (in the second administration: 62 items) corresponding to items of the "CSMS Mathematics Test" by the National Foundation for Educational Research (3), to the "Assessment of Rational Number Concepts" and the "Assessment of Rational Number Relationship" developed by the National Science Foundation (6),(7) and to items in different school-books. These items only test the understanding of the rational number concept, not the skill in dealing with calculations. With only a few exceptions we considered only positive rational numbers smaller than 1 with denominators 2,4,8,3,6,5,10,(7),12. We took care to present the problems in contextual frameworks or sketches. Items consisting merely of numerals and operations ($\frac{4}{6}$ of $\frac{4}{1}$), were only inserted as means for control. Difficulties in understanding the text were minimized by a clear and simple style. Auxiliary lines and hints were sometimes given to make the solution definite. The items were chosen in a way that the test can be given to the grades 6-10. The test did not include decimals and percentages. For some items we used the multiple-choice method. All the items were categorized as shown in fig. 2:

fig. 2



Find the correct solution:

A is larger than B ☐

A is smaller than B ☐

A is equivalent to B ☐

A_2 B_{1b} E_1 T_2 V_2 S_3

item 4

Mr Brummer earns 2400 DM per month. He spends $\frac{1}{6}$ of his salary on the rent for his house.

How much is the rent ?

A_3 B_2 E_1 T_1 V_2 S_1

item 13

When constructing the test we took care to consider all significant combinations of categories.

The adminitration of the test

In order to prove our hypothetical scheme of categories we tested it

a) in October 1984 with 90 pupils of the 9th grade in a comprehensive school in

Berlin-Neukölln

b) in April 1985 with 218 pupils also of the 9th grade in a comprehensive school in Berlin-Spandau

c) in September 1985 with 70 pupils of the 9th grade in a Real-Schule in Wentorf (Schleswig-Holstein).

The test was devided into two subtests, so that in each subtest there were corresponding items - items of the same categories and of the same assumed difficulty. For each subtest the pupils were given the time of one lesson (45 minutes).
The items were handed out in random order, different for each pupil - so as to avoid any effects of learning. The subject matter (rational numbers) had deliberately not been repeated in the classroom before the administration of the test. The pupils were allowed to use a ruler with scale. Time limit and pressure on the pupils were avoided as far as possible.

The evaluation of the test

The items were classified as correctly solved $\{0\}$ or as incorrectly or not solved $\{1\}$. For the analysis of the data we used different methods of "cluster analysis" (10) and "multidimensional scaling" (11).
This is the matrix of data for the second administration (3 items were not considered because too many pupils hadn't solved them):

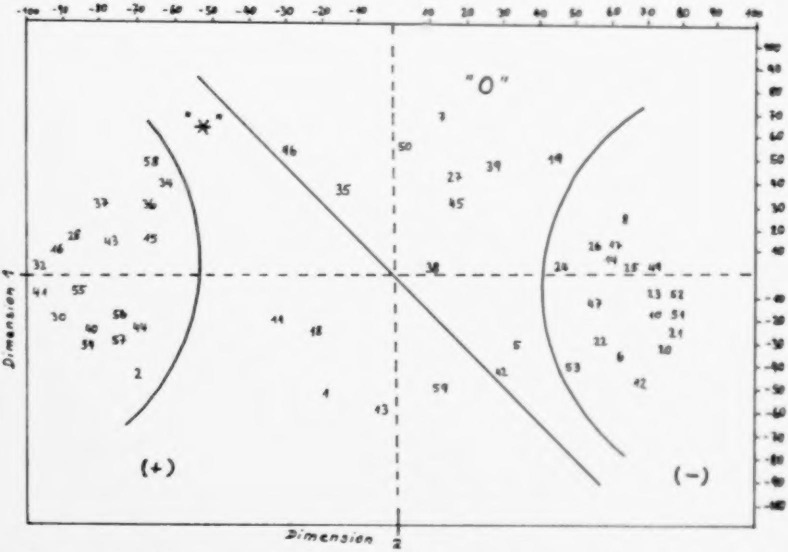
		item								
		1	2	3	.	.	.	57	58	59
pupil	1	0	1	1				1	1	0
	2	1	0	1				1	1	1
	3									
	.									
	.									
	.									
	219									

By cluster analysis (Ward, Complete-Linkage) we got 3 clusters of pupils. For each cluster there was a similarity concerning the correct or incorrect solution of the items. We considered the solutions of the items for each cluster independently. We compared the solution (0 or 1) of all possible pairs if items for each pupil of this cluster. If there was a correspondence (both items solved correctly or both items solved incorrectly or unsolved) the pair got the index 0, otherwise the index 1. This analysis was carried out for all pupils. Considering all the pupils now we added up the non-conformity for each pair of items (a_{ij} with $a_{ij}=a_{ji}$). We thus got a symmetric matrix of "non-conformity" of the items. This matrix was interpreted as a matrix of "dissimilarities" of the item.

$$\begin{array}{ll|llllll}
 & & \text{item} & & & & & \\
 & & 1 & 2 & 3 & 4 & \cdots & 59 \\ \hline
\text{item} & 1 & 0 & a_{12} & a_{13} & a_{14} & & \\
 & 2 & a_{21} & 0 & a_{23} & a_{24} & \cdots & \\
 & 3 & a_{31} & a_{32} & 0 & a_{34} & & \\
 & \vdots & \cdots & & & & \ddots & \\
 & 59 & & & & & & 0
\end{array}$$

The procedure of multidimensional scaling (MDS, MINISSA) calculates the dissimilarities as distances and illustrates the items as points in a space with n dimensions.

Example:



Results and their interpretations will be given in the presentation and will be a good foundament for discussion.

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COUNTING PROCEDURES USED BY KINDERGARTEN CHILDREN

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Fuson, Richards & Briars (1982) have investigated systematically the acquisition of the number word sequence by young children aged 3.6 to 5.11. For those aged 5.0-5.5 (n=10) they report that without objects, they could count up to an average of 40.19 and for rows of blocks they could count up to 22.38 . For children aged 5.6-5.11 (n=10), these figures were 38.17 and 25.00 respectively. These results indicate clearly that by the time they reach elementary school, most children have acquired considerable counting skills. However, these do not necessarily reflect their understanding of the counting procedure. The counting procedure was studied extensively by Gelman & Gallistel (1978) who identified three "how to count" underlying principles, the one-one principle, the stable order principle, and the cardinal principle. In analyzing the different kinds of possible errors occurring in the application of the one-one principle, they suggested three distinct classes: "tagging" errors, partitioning errors, coordination errors. Their observations provide us with the following data for 5-year-olds:

			One-one errors		
Set size	No of counts	% errors	Tag-duplication	Partitioning	Coordination
7-19	207	35.7	1.4	23.2	19.8

(a count trial could have more than one error)

	Partitioning Errors				Coordination Errors			
Set size	Double count	Recount	Omit	Stop too soon	Beginning	End	Overrun	Asynchrony
7-19	7.3	1.5	12.1	2.3	0	14.0	4.8	1.0

These results indicate that there are almost no tag-duplication errors (1.4%) at the age of 5 and that most errors are of the partitioning

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(23.2%) and coordination (19.2%) types. The majority of coordination errors (14%) occur because the child missed or double-counted the last item. It seems to us that this type of error could easily be considered as being a partition error since whether it occurs at the very end or in the middle of the count, in both cases it reflects the inability to keep track of the objects enumerated. Thus we can consider the partition errors to be by far the predominant ones.

While the above studies reveal how difficult it is for children to master the skill of partitioning, they do not inform us about different partitioning procedures or the frequency with which they are used as a way of separating the objects remaining to be counted from those already counted. In some exploratory work, we found that kindergartners have three distinct spontaneous counting procedures: visual counting (V) in which the child keeps track of the objects visually without any physical contact; touch-counting (T) where each object is touched without any displacement; physical partitioning (P) where the child separates the objects as they are counted by pushing them aside one at a time. In the first two procedures, the partition is established mentally and this can lead to many errors, especially when the objects are displayed randomly.

The frequency of these partitioning procedures was but one of the questions we wished to investigate. We also wondered if children in this age group would perceive one of the procedures as being better, would prefer one in particular, would choose a specific one "if they had to teach a child who does not yet know how to count", might remember how they themselves had learned. Finally, we wondered if inducing this kind of reflection in them would affect in any way their counting procedure.

To find answers to our questions we interviewed 31 kindergartners (16 girls, 15 boys considered weak, average or strong by their teacher) in 5 different French schools of Greater Montreal (one in upper and four in middle class neighborhoods). The interviews were conducted by 15 prospective elementary school teachers who were in the second year of their B.Ed. program, and as such, had enrolled in a course on the teaching of arithmetic at the primary level. These 15 future teachers were grouped into small teams (from 2 to 5). Their training consisted of various simulations, the study of video-interviews and the study of the semi-standardized questionnaire to be used in the experiment. Each interview was handled by two team members, one interviewing, the other one observing and audio-recording. Each recording was then totally transcribed.

We first decided to assess the children's knowledge of the number word sequence by asking them to count out loud, and then, their ability to enumerate, which was tested by determining how far they could count when given a pile of 100 chips. The following table describes the counting scope of 29 subjects.

Age	N	No object counting	Counting pile of chips
under 6*	14	38.1	36.7
S.D.		26.5	26.9
Range		(16-119)	(16-119)
6.0-6.6	15	54.2	43.1
S.D.		41.1	23.5
Range		(27-179)	(15-100)

Two subjects,who could only count up to 3 and 9 respectively, were eliminated since they were judged to have too many difficulties with their number word sequence to provide us with valuable information regarding their understanding of the one-one principle. If we compare our results with those of Fuson et al, we notice that for their 5-year-olds, their knowledge of the number word sequence is almost the same as that of our subjects in the under-6 age group, while the difference in counting with objects might be attributed to the way in which our respective tests were designed. We have found that the difference between the knowledge of the number word sequence and the counting of objects is relatively small in this age group (38.1 vs 36.7). It is interesting to note that with three of our subjects, their counting of objects went beyond their initial verbal recitation of the number words.

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* 4 children were between 5.2 and 5.5, and 10 between 5.6 and 5.11

We tested the notion of cardinality explicitly by asking how many blocks were in a set of 15 cubes laid out at random in front of the child. Everyone spontaneously set out to enumerate the given set to answer the question, thus demonstrating their grasp of this notion. We noted that 14 subjects used a physical partition (P), 11 used touch-counting (T) and 4 resorted to visual counting (V). It is interesting that among the 6 children who made a mistake, 5 used T, while only 1 had used P. The two types of errors observed were double-counting (3) and loosing track (5), some mistakes involving two types of error. The rate of errors we observed (20.7%) is much lower than that found by Gelman & Gallistel (35.7%). But it is difficult to compare since half of our subjects were over the age of 6.

We also verified explicitly whether or not children perceived that the result of enumerating a given set had to be unique. Ginsburg (1977) had shown that at an early age, some children did not find any contradiction in obtaining different results from different counts of the same set. Only one of our subjects thought that two different results would be acceptable. In addition, we investigated how children understood the one-one principle by telling them that we were going to commit counting errors in front of them and asking them to spot our mistakes. One mistake was counting twice the same chip and the other mistake was to skip one chip. Everyone of our 29 subjects was able to identify the two mistakes,except the subject mentioned above who did not spot the double-count, in spite of the fact that he knew his number word sequence up to 59 and could count a pile of chips up to 49.

As mentioned earlier, we wished to find out how the three counting procedures were perceived by the children. We provided them with photographs of three children, each one using a distinct procedure

V: Visual T: Touching only P:Physical partition

and in each case, the interviewer acted out these procedures. The photographs were presented with each question as a reminder of the three procedures. The questions were (a) "Do you think there is a way of counting which is best?"; (b) "When you count, which way do you prefer?"; (c) "I know a child who does not yet know how to count. If I asked you to show him how to count, which way would you pick to show him?"; (d) "When you learned how to count, do you remember which way?". The following table gives the frequency of their first choice.

	Under 6 n=14				6.0-6.6 n=15				Total n=29			
Question	P	T	V	?	P	T	V	?	P	T	V	?
(a) best	6	6	2	-	9	2	4	-	15	8	6	-
(b) preferred	9	2	3	-	10	3	2	-	19	5	5	-
(c) to teach	7	2	4	1	9	5	1	-	16	7	5	1
(d) learned	5	2	1	6	10	2	1	2	15	4	2	8

Overall, we note that in considering what is the best procedure, about half chose P while the others chose a non physical partition. This is comparable to the ratios found in the spontaneous procedures mentioned earlier (14P, 11+4=15 others). What seemed evident is that the questions we raised were totally new for the children and that probably they may not have had sufficient time to reflect about them. This might explain the shift we note in their choices for the preferred procedure. As far as preference is concerned, there does not seem to be any marked difference between the age groups. But when it comes to consider teaching, while P remains the principal choice for both groups, only one child among the older ones favors V. Fewer of the younger children seem to remember how they learned how to count. Perhaps the ability to answer such a question has to do with maturation.

By way of conclusion, we wish to present data regarding the last task in which we simply asked the children to find how many blocks there were in a given pile of cubes (10). The frequency of their spontaneous choices (P=15, T=8 , V=6) does not seem to differ much from that found in the initial task (P=14, T=11, V=4). But when the fourteen children who had used T or V were asked "Would you have another way of counting to make sure that there are ten blocks?", eleven of them answered this question by counting again using a physical partition. These results seem to indicate that even if there was no direct teaching involved in the experiment, the mere inducement to reflect on various counting procedures brings to the child a greater awareness of the different choices available and the relative merits of physical partitioning.

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Children's conceptions of multiplication Neil MacCuish

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INTRODUCTION

According to Piaget the source of mathematical structures during development is the child's own activity. That is, mathematical knowledge arises through the child reflectively abstracting some principle of his/her own functioning. Multiplication seems to be an intriguing area for investigation of reflective abstraction. It involves the use of an operation on an operation. And the higher level operation can be thought of as a copy of the lower level operation. That is, in 5 x 3 not only is there the repeated addition of three but also the five can be broken up and composed additively through the distributive laws.

In the present study the subjects were given a multiplication task which could be solved by simple addition. The subjects were given a completed sum as a cue, for example 261 x 20 = 5220, and then given a probe question derived from the cue by adding or subtracting one from the multiplier, for example 262 x 20 = ? The way the subjects utilize the information given in the cue is informative of their conceptions of multiplication.

METHOD

Subjects There were thirty eight subjects. The subjects formed one third year class and one fourth year class (age range 9.6.7 - 11.5.14).

Procedure The subjects were tested in their classroom as a group. The experimenter wrote up the cue as a completed sum on the blackboard so all the subjects could see it; underneath he wrote the probe question. After the main task the subjects were given the six control questions.

Material The cue and the probe questions are shown in table 1. There are four types used. The multiplier can either be increased or decreased by one, giving, in effect addition or subtraction. And the multiplicand is either a multiple of ten or simply units. The control questions are addition or subtraction sums comparable in difficulty to the sums implicitly defined by the multiplication task.

TABLE 1 : CUE AND PROBE QUESTIONS

Actual order presented	Type of implicit operation	Tens or units	
1	Add	Tens	261 x 20 = 5220 262 x 20 =
8			30 x 322 = 9660 30 x 323 =
3	Subtract	Tens	312 x 20 = 311 x 20 =
5			30 x 255 = 7650 30 x 254 =
4	Add	Units	716 x 4 = 2864 717 x 4 =
7			3 x 715 = 2145 3 x 716 =
2	Subtract	Units	3 x 339 = 1017 3 x 338 =
6			564 x 4 = 2256 563 x 4 =

TABLE 2 : CONTROL QUESTIONS

Actual order presented	Question
9	7140 - 20
10	5130 + 30
11	3 + 3155
12	5118 - 4
13	8140 - 20
14	1960 + 30

TABLE 3 : RESPONSES FOR EACH TYPE OF QUESTION (IN PERCENT)

Subjects' Response	Question Type:-		Question Requires:-	
	Addition of ten	Subtraction of ten	Addition of units	Subtraction of units
Correct	22.37	19.74	17.11	15.79
Add 1	22.37	3.95	27.63	6.58
Add 10	13.16	1.32	2.64	1.32
Subtract 1	3.95	19.74	5.26	26.32
Subtract 10	0	15.79	0	2.64
Other error	38.16	39.47	47.37	47.37

RESULTS

1. Thirty five percent of the error total was the addition or subtraction of a unit, and eleven percent was the addition or subtraction of a ten, so that forty six percent of the error total was of this sort.

2a) Subjects gave significantly more of the error response involving plus/minus ten on the questions which had a multiplicand which was a multiple of ten than on questions which had a unit multiplier (Sign test $p < .002$).

b) Conversely subjects gave more errors involving plus/minus one on questions in which the multiplicand was in units than on questions in which it was a multiple of ten (Sign test $p < .08$).

3a) The subjects gave significantly more plus one/ten responses on questions in which the multiplier was increased by one than on questions in which it was decreased (Sign test on third years $p < .001$, on fourth years $p < .001$).

b) Conversely, the subjects gave more minus one/ten responses on questions in which the multiplier was decreased by one than on questions in which it was increased (Sign test on third years $p < .001$, on fourth years $p < .05$).

4. The subjects were significantly better at the control questions than at

the comparable multiplication tasks (Twelve sign test all significant : eight at the .001 level, two at the .01 level, and two at the .05 level).

5. In the third year group the subjects who scored zero on the control questions gave significantly less of the error under consideration (Mann Whitney $z = 2.22$, $p < .05$).

DISCUSSION

As would be expected the subjects were worse on the multiplication task than on the comparable addition or subtraction question. Although not surprising this does indicate that the probe and cue technique did bring into play the subject's conceptions of multiplication. Over a third of the responses were the addition or subtraction of a ten or unit. For comparison only approximately a fifth of the responses were correct. Such a common error pattern must reflect the subject's conception of multiplication.

Piaget found almost exactly this error in his study of the genesis of proportionality (Piaget 1977). Both the subject and the experimenter had a similar stripe with a sequence of holes on it and the subject's task was to place his/her marker twice as far as the experimenter placed his, that is to place the marker at 2N to correspond to the experimenter's marker at n. Piaget found a stage 1 in which the subject placed the marker at n + 1 instead of at 2n. He also found a stage 2 in which the subjects placed the marker at n + k, where k is bigger than 1 but less than n. He found similar responses in another task in which the subject was shown three different length eels and told to provide food in proportion to the eel's length. The additive strategy is not only found in these two tasks but it has been found in a variety of proportionality tasks.

In the present task in making the error under discussion subjects showed a differential sensitivity to the operation and the multiplicand. That is on questions in which the multiplicand was a multiple of ten the subjects were more likely to give a plus or minus ten response than on questions in which the multiplicand was in units; conversely they were more likely to give a plus or minus one response on questions in which the multiplicand was in units. And on questions which implicitly involved addition the subjects were more likely to give a plus response than on questions implicitly involving subtraction; conversely on questions which implicitly involved subtractions subjects were more likely to give a minus response.

Multiplication is a multi- componented task. Obviously children making these errors have not completely acquired the complete structure, but this differential sensitivity shows that they have some partial knowledge of multiplication. They know both that multiplication involves repetition (as either repeated increase or decrease) and that it involves repetition of a definite something. This partial knowledge can be described as an intuition about multiplication. That is, using intuition in a sense analogous to Brown and McNeil's (1966) tip-of-the-tongue phenomena; they showed that if subjects thought they knew a word but were unable to recall it then the subjects possessed some accurate partial knowledge, for example above chance level knowledge of the first letter.

Piaget believed that the reason for the plus type responses in his tasks was that the primacy of one way actions in the child's thought leads the child to be dominated by order effects. So long as the order is preserved, which it is by the plus type responses, the child is satisfied. Piaget is thus locating the reason for this error in the overall organization of the child's thought.

I would like to raise the possibility that the child's difficulty has a more circumscribed origin, namely in the child's representation of the task. Karmiloff-Smith (1984) argues that the same three phases of development of representational processes can be found at all levels of development. Along these lines it is suggested that the additive error in the present task is very akin to the pre-concept which Piaget described in the much younger child. In the pre-concept a part stands for the class and can substitute for it instead of merely representing it; it is a privileged sample to which other elements are assimilated.

In the present task the child is dealing with what is implicitly an infinite set of repeated additions. Part of the formation of multiplicative structures is the collapsing of that infinite set of procedures of addition to an explicit representation. Prior to the formation of this meta-level representation the child confuses the member with the set and selects one procedure to represent the whole set of procedures. Both the pre-concept and the present error arise from the same cause, namely the lack of explicit meta-level representations, but in the present case operating on a more advanced representational level, that is on a problem space of a set of procedures instead of a set of perceived objects.

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Why some children fail to learn the four rules of Arithmetics

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Earlier investigations have shown that about 15% of each age cohort in Sweden fail to acquire sufficient mastery of elementary skills in Arithmetic. On the assumption that this is due to the lack of the conceptual prerequisits for the formal teaching adopted in schools 105 school beginners (7 years olds) were interviewed in order to find out the sense in which they use counting words, how they conceptualize numbers and how they approach simple arithmetic problems. Some of the findings were:

1. Counting words are used in a number of qualitatively different ways by children. They are understood as denoting

   sounds,
   movements,
   extent,
   names, etc.

2. There are two distinctly different ways in which children approach simple addition and subtraction problems, namely

   by counting, or
   by structuring.

   The former means that they are counting one object (e.g. a finger) at a time while the latter refers to the fact that they see relations (part-whole relations in particular) within and between numbers. There are several different strategies within both approaches.

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3. Understanding the fact that counting words have a number meaning is a necessary prerequisit for the development of the second approach (i.e. structuring).

4. "The structuring approach" is a necessary prerequisit for the development of *elementary number concepts* (i.e. the mastery of all the possible relations within and between the numbers 1-10).

5. The development of "the elementary number concepts" is a necessary prerequisit for the acquisition of further arithmetic skills (addition and subtraction over 10, multiplication, division).

In consequence with these findings,an experimental teaching program was set up, primarly aimed at developing the number meaning of counting words and subsequently at developing the elementary number concepts. 24 school beginners were judged to lack the necessary prerequisits for acquiring basic skills in Arithmetic. 12 participated in the experimental program carried out in two different classes while 12 remained in 5 different classes with regular teaching. *All* the children in the experimental group acquired *all* the skills in Arithmetic they were supposed to acquire during the first two school years according to the curriculum, while at least 6 of the 12 children in the control group seemed to have serious difficulties in Mathematics.

PRESERVICE TEACHERS' CHOICE OF OPERATION FOR MULTIPLICATION AND DIVISION WORD PROBLEMS

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In recent years much has been written about children's and adolescents' misconceptions concerning the operation needed to solve multiplication and division word problems (e.g., Vergnaud, 1980; Hart, 1981; Bell, 1982). The purpose of this study was to explore whether preservice elementary teachers have the same misconceptions noted among elementary and secondary school children.

Bell categorizes the errors eleven-year-old British students make in choosing an operation for word problems that involve decimals. The categories he describes are (1) misinterpretation of decimal numbers, (2) over-generalization of rules beyond their domain of truth, (3) detachment of meaning from symbols, and (4) misinterpretation of potentially distracting cue words or contexts.

Fischbein, Deri, Nello, and Marino (1985) suggest that primitive models may account for the errors adolescents make in selecting the appropriate operation for word problems. They argue that the arithmetical operations generally remain linked to implicit, primitive, behavioral models that influence the learner's choice of operation even after formal algorithmic training. The primitive models obey constraints, imposed by their behavioral nature, that do not always match the contraints on the formal

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mathematical operations. Fischbein et al.'s findings confirm that when contradictions exist between the implicit roles of the primitive model and the multiplication or division word problem being solved, the primitive model may slow down, divert or even block the student's solution.

In the present study, an attempt was made to check the assumption that the primitive behavioral models described in Fischbein et al.'s study also affect adults' choice of operation for solving simple multiplication and division word problems. Furthermore, the authors intended to document other multiplication and division misunderstandings of preservice teachers.

METHOD

The subjects were 129 female college students enrolled in one of the math content or methods courses for early childhood education majors in a large university in the United States.

The 26 multiplication and division word problems used in Fischbein et al.'s study were modified in an attempt to use terminology and notation more familiar to the American population. These 26 items were distributed between two test forms. Each student completed six multiplication, seven division, and five addition or subtraction "filler" items. The subjects were instructed to write an expression that would lead to the solution of each problem, but they were asked not to compute the solution.

Thirty-three students from two of the four participating classes were interviewed. In the interviews, students were given problems similar to those they missed and asked to write an expression that could be used to solve each problem. They were asked to explain why they wrote the expression they did and how they would check their work. After discovering that an answer was incorrect, they were asked to verbalize what led them to write the wrong expression. Finally the interviewer would substitute into the word problem whole numbers that conform to the constraints of one of the primitive models and ask the student if this helped her write the correct expression for the original word problem.

RESULTS

The authors found that students in the preservice teacher population are influenced by the same primitive, behavioral models for multiplication and division as are students in Fischbein et al.'s 11-15 year-old population. For example, a primitive multiplication model is that of repeated addition. When this concept of multiplication prevails, the operator (or number of sets) "must" be a whole number. As an example of the influence of the repeated addition model, data are presented for the preservice teachers' success with two word problems. One word problem was "For one cake you need 2.25 grams of spice. How much spice do you need for 15 cakes?" A correct expression for the solution of this problem was provided by 99% of the students. A similar word problem but with a decimal operator was "For one kilogram of meatloaf you use 15 grams of salt. How much salt do you need for 1.25 kilograms of meatloaf?" Only 72% of the students wrote a correct expression. Results from two other problems show the influence of the primitive, partitive division model. This model "requires" that the divisor be less than the dividend. Eighty-nine percent of the students wrote a correct expression for the problem: "In 8 boxes there are 96 cartons of milk. How many cartons are in each box?" However, only 34% of the same students wrote a correct expression for the problem: "12 friends together bought 5 pounds of cookies. How many pounds did each one get if they each got the same amount?" Furthermore, 42% of the students wrote the incorrect expression, 12 ÷ 5, for this problem. Thus, a substantial portion of the preservice teachers reversed the role of the divisor and dividend writing expressions that did not match the problem but did match the constraints of the primitive partitive model.

One of the misconceptions noted among the preservice teachers was an over-generalization of the procedures used with unit fractions. Interviewees believed that since 6 x 1/2 can be computed as 6 ÷ 2, 900 x .75 can be computed as 900 ÷ .75! Unlike many of the other misconceptions noted, this error does not appear to be widely discussed in the literature on children's errors with decimal operations.

Although the overall results on the written instrument, shown in Table 1 below, might lead the reader to conclude that the misconceptions were held by only a small portion of the preservice teachers, our interview data suggest otherwise. Every interviewee, including those who had made only one error on the written work, gave evidence of holding at least one misconception. The preservice teachers were not able to describe their thought processes with great clarity. A large portion of the interviewees explained their errors by saying "Word problems confuse me," "Decimals throw me out," or "The units confuse me." Thus, the interviews led the authors to believe that almost all of the preservice teachers were influenced, usually unconsciously, by erroneous beliefs about multiplication and division.

Another phenomenon observed in the interviews is of interest. A substantial number of those students who said that a given word problem involving decimals could be solved with one operation (e.g., division) claimed that when whole numbers were substituted for the decimals in the problem, another operation (e.g., multiplication) was needed. Furthermore, these students did not see any contradiction between these claims.

Table 1

Number (Percent) of Respondents by Score*

Number of Items Correct	Form A n = 64	Form B n = 65	Total
13	6(9)	10(15)	16(12)
12	6(9)	11(17)	17(13)
11	16(25)	13(20)	29(23)
10	8(13)	9(14)	17(13)
9	11(17)	11(17)	22(17)
8	5(8)	2(3)	7(5)
7	6(9)	3(5)	9(7)
6	3(5)	4(6)	7(5)
5 or less	3(5)	2(3)	5(4)

*Only the 13 multiplication and division items are included.

IMPLICATIONS

The number of preservice teachers experiencing difficulties with the selection of multiplication and division operations for word problems suggests that they might have other, more basic, misunderstandings. In fact, the interviews confirmed that some of the preservice teachers' difficulties were related to their misunderstanding of the notation for decimals and division. An investigation of their understanding of these and other fundamental notions might be appropriate. Another interesting investigation would be to determine the extent to which pupil materials contain only problems that conform to the primitive models. Research could also be designed to answer other questions related to the influence of the primitive models, for example, facility with measurement versus partitive examples.

The results of this study indicate that efficient strategies must be developed for training teachers to monitor and control the impact that misconceptions and primitive models have on their own thinking and their students' thinking.

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Early Addition Story Problem Performance: How Does It Relate to Schooling and Conservation?

by

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Previous researchers reported significant differences between the problem solving achievement of first graders who conserve number and those who do not conserve. Researchers also found that problems which involved transformations were significantly more difficult than those which did not involve transformations (Steffe, 1967; LeBlanc, 1968). A transformation is defined as an implied physical movement of objects, e.g., "Mary has three dolls and her mother gives her two more." No transformation is a static condition, e.g., "John has two frogs and David has three frogs."

An important question has remained unanswered. Since Steffe's addition study was executed on a population completing the first grade, to what extent was the addition transformation finding related to the developmental level of the subjects and to what extent was it due to history, particularly first grade instruction? This is a particularly important question since most first grade teachers model addition as a transformational process.

A Conflict. As Piaget (1965) noted, children are so bound by perception that they believe quantitative changes occur which do not; they only appear so; given five coins arranged in unequally dispersed arrangements, the young child believes one set is more than the other. Because of perceptual data, in the study conducted by Van Engen and Steffe (1966) children thought $2 + 3 > 5$. Judged on the basis of this perceptual confusion, it appears that early experiences with both addition and equality would best be served if addition could be modeled for preconservers without transformations (movement). Thus, in a given physical experience, $2 + 3$ would "look like" and, hence, be thought of as another name for 5. Two critical questions exist: (1) Can non-transformational addition be readily learned by non-conservers? (2) Were earlier results influenced by first grade instruction?

To answer the first question without the influence of the second, data were collected on kindergarten children who had received no formal addition instruction. To answer the second question, modeling tasks were created to determine if first graders modeled addition with transformations even when the problems were non-transformational.

Procedures and Materials

Thirty-six kindergarten and thirty-six first grade children were randomly selected from one Area of a large school district which contained a good cross-section of the city's ethnic and socio-economic groups.

In the spring, Test of Quantitative Comparisons (TQC) (Harper and Steffe, 1968) and a four item addition test were individually administered to all subjects; the KR-20 reliability of the TQC is .86 (Steffe and Johnson, 1971). The addition test consists of four items, two transformational (T) and two non-transformational (NT). The order of the questions was determined randomly for each subject. Modeling reliabilities were: all addition, .99; transformational, .99; and non-transformational, .92. For accuracy they were: all addition, .86; transformational, .60; and non-transformational, .83.

The researcher and subject sat across a table from one another. In front of the subject was a small basket of chips and two eight inch cardboard dolls. At the beginning of the addition test, each subject named the dolls, e.g., Michelle and Ramos, and the chips, e.g., hamburgers. All four questions then used the names and label provided by the subject. The following pairs of addends were used in the four questions: (4,2), (2,5), (4,3), (3,5). Each question was "chunked," i.e., presented one-sentence-at-a-time with a pause during which the subject was required to "act out" the sentence with the manipulatives provided. Each subject received 0 or 1 point for accuracy of each response; the criterion was correct response ±1. Each subject received 0, 1, or 2 points for the modeling behaviors exhibited relative to each problem according to the following criteria:

	Transformational	Non-Transformational
1 point	Count first set of chips and place in front of doll A.	Count first set of chips and place in front of doll A.
1 point	Count second set of chips and place in front of same doll, A.	Count second set of chips and place in front of doll B. (No point if the two sets were combined.)

Analyses and Results

Initial multiple linear regression analysis of the dependent variables using sex and age (in months) as criterion variables confirmed that variance due to age must be statistically removed from subsequent analyses. Likewise, a systematic, although non-significant sex bias was statistically controlled.

First, a nested analyses of covariance of the modeling scores was completed.

The subjects' abilities to model transformational addition was affected by their abilities to make quantitative comparisons (p = .030) and the effect of schooling upon their non-transformational addition modeling scores was significant at the .057 level.

Then a nested analysis of covariance of the accuracy scores was completed. The significance within the addition accuracy scores was found to be attributable to the TQC effect on both transformational (p = .004) and non-transformational (p = .008) problems. The effect of schooling upon addition accuracy scores was also significant on both transformational (p = .002) and nontransformational (p = .006) problems.

To interpret the results of the analyses of covariance, selected data were examined with t tests. There were no differences in performance on transformational and non-transformational accuracy among first graders at the .05 level. A difference was found among kindergarteners only when a criterion of 2/2 was used on each problem type. However, the difference did not hold up when a criterion of 1/2 on each problem type was used; this criterion level is probably more appropriate given (1) the higher incidence of counting errors among kindergarten children, and (2) the open-ended nature of the questions.

Modeling scores of all subjects were also examined in more detail. There is a difference between kindergarten conservers and non-conservers when a 2/2 success level is used but not when a 1/2 success level is used, and during the first grade, there are no differences in modeling of transformational and non-transformational problems by non-conservers when a success level of 2/2 is applied, but they are better at transformational when a 1/2 success level is used. And, there is a difference between problem types among first graders who are conservers when a success level of 2/2 is used but not when 1/2 is used.

Discussion

Question 1: Can non-transformational addition be learned by non-conservers?

Under the presentation conditions used, first graders' accuracy is similar using both success levels, and kindergarteners' accuracy is similar when a 1/2 success level is used. Further, 50% of the non-conserving kindergarteners can successfully answer non-transformational questions and 60% can successfully answer transformational questions under the conditions used in this study, a difference of only 10%; and 79% of the non-conserving first graders could answer non-transformational questions correctly. From these data, one can argue rather forcefully that under these problem presentation conditions non-transformational addition problems are not more difficult than transformational questions for non-conservers. It should be noted as well that non-conservers improved substantially in their success on non-transformational questions whether evaluated on a 2/2 success criterion ($\overline{x} = .50$ to $\overline{x} = .64$) or a 1/2 success criterion ($\overline{x} = .50$ to $\overline{x} = .79$). There seems to be no reason to exclude non-transformational addition from the curriculum designed for non-conservers.

Question 2: Were earlier research results influenced by first grade instruction?

The results of this study obviously differ from those of Steffe (1967) and LeBlanc (1968). Why? The initial concern regarding their research studies was prompted by the realization that their data were collected late in the first grade. The accuracy results of the present study indicate that differences found by Steffe were not supported in the present study. This may be due to (1) required rather than optional use of manipulatives, (2) individual rather than group administrations, or (3) "chunking" rather than "whole" oral problem presentation. However, the modeling data indicate support for the original conjecture that differences observed by Steffe may have been magnified by the instructional behaviors of first grade teachers. This conjecture can be explored by examining the modeling behaviors.

There were modeling differences for

Kindergarten non-conservers under a 2/2 success criterion;
First grader conservers under a 2/2 success criterion; and
First grade non-conservers under a 1/2 success criterion.

Using a liberal criterion (1/2), only first grade non-conservers performed differentially. During the data collection phase of this study, they frequently joined the two sets in non-transformational problems. When using a conservative criterion (2/2), only the most unsophisticated (kindergarten non-conservers) and most sophisticated (first grade conservers) exhibited differential performances. Given that counting errors at this level are highly and positively correlated with sophistication, it seems reasonable to downplay the importance of the kindergarten differences and to focus on the performances of first graders. Once again, it was observed that first grade conservers joined sets inappropriately in non-tranformational questions.

One fairly obvious interpretation of the preceding results is to confirm, at least in part, that the Steffe results were influenced by first grade instruction. By the end of grade one, children seem inclined to view addition as a transformational activity even when problems do not involve explicit transformations; therefore, it seems reasonable to assume that by the end of the year first graders perform better on transformational problems.

Conclusion

There appear to be three conclusions which can be made based on this study:

1) Using the problem conditions described in this study, transformational addition is no easier than non-transformational addition for kindergarten and first grade children when controlling history (schooling).
2) Non-conservers can learn to model and accurately solve simple non-transformational addition problems.
3) Teachers should do one of the following when teaching addition story problems to non-conservers:
   - A. Use only non-transformational addition, or
   - B. Use physical transformational models with implied rather than real transformations, i.e., no movement of the two sets once they have been positioned.

Table 1 **Modeling means and standard deviations on a one-point scale, and t tests using a one of two criterion**

	Grade Level			
	Kindergarten		First	
	Non-conserv	Conserv	Non-conserv	Conserv
Non-transformational	$\bar{X}$ = .85 SD = .37	$\bar{X}$ = .88 SD = .34	$\bar{X}$ = .71 SD = .47	$\bar{X}$ = .91 SD = .29
Transformational	$\bar{X}$ = .90 SD = .31	$\bar{X}$ = .94 SD = .45	$\bar{X}$ =1.00 SD = .00	$\bar{X}$ =1.00 SD = .00

Table 2 Modeling means and standard deviations on a one-point scale, and t tests using a two of two criterion

	Grade Level			
	Kindergarten		First	
	Non-conserv	Conserv	Non-conserv	Conserv
Non-transformational	$\bar{X}$ = .50 SD = .31	$\bar{X}$ = .69 SD = .48	$\bar{X}$ = .71 SD = .47	$\bar{X}$ = .73 SD = .46
Transformational	$\bar{X}$ = .90 SD = .31	$\bar{X}$ = .75 SD = .43	$\bar{X}$ = .93 SD = .27	$\bar{X}$ =1.00 SD = .00
	t = -2.99 p = .005	t = -.36 p = .29	t = -1.38 p = .10	t= -2.81 p =.005

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IS 89 LARGER THAN 91?

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Introduction

One of the most common addition errors is the one in which no trading (regrouping) takes place. For example, when the two addends are 148 and 236 the sum will be either 374 or 3714 or something like it.

A comparable error with subtraction is one in which the top digit is subtracted from the bottom digit. In this case again no trading takes place (cf. Fuson 1985, Becker 1985).

These kind of problems are mostly presented in the third grade. The almost universal practice is doling out one more place in addition and subtraction in each successive grade (1-digit in first grade, 2-digits in second grade, 3-digits in third grade etc.) As long as pupils do not understand when and how to trade with addition and subtraction problems, these errors keep coming up regardless of the number of digits used in the problem. This also means that these errors do not automatically stop after the second or third grade. It is therefore necessary to tackle these errors early in school and efficiently.

Fuson (1985) suggests in preventing these errors, to teach the place value system together with the addition and subtraction algorithm and not before then.

We decided it is more efficient to teach the place value system before introducing the algorithm. And of course there are a number of reasons for this decision. To understand when and how to trade presupposes a body of knowledge that is more complex than we usually think. In fact three abilities are involved: (a) the ability to generate number names according to our number system, (b) the ability to understand the number meanings represented by the number names and (c) the ability to use our place value notation system (Carraher 1985). Children generally master the ability mentioned under (a) just before or in the course of the first grade. It is the abilities (b) and (c) that matter when constructing a teaching program. With respect to (c), when studying our place value system there are two aspects to distinguish. Firstly, our system is a ten base system, i.e. it is a system with base ten in which ten ones are represented by one ten, ten tens are represented by one hundred etc. Secondly, our system is place holding, i.e. a place in a number has a given value, known as place value. For example the one in 41 has a different value than the one in 14. The one in 41 represents a one and in 14 a ten, in 144 a hundred etc. Our numeration system, to be precise, is a ten-based place holding system; it is the Hindu-Arabic numeration system.

When we ask a second grader to write down what number comes after 55, he often writes down 65. Or he writes 69 as an answer to the question:"What number comes before 79?". Or he circles the first number of the numbers 89 and 91 when asked for the larger number. These are very well-known errors and according to our data about 50% of second graders make these errors regularly. With some children they disappear gradually; with others they "transform" into persistent errors as mentioned in fig. 1.

Young children very often do count very well up to a hundred or somtimes a thousand without really 'knowing the structure behind it. In other words they are able to generate number names according to our number system (the above mentioned ability a) without being aware of the number meanings and the underlying place holding system (the abilities b and c).

Writing down the numbers and in this way confronting the children with the notation system makes the children aware of the properties of our number system.That is why, in our opinion, we have to teach the properties of our number system, i.e. base ten and place value together with the notation of the known number names and before introducing the addition and subtraction algorithm.

The development of our numeration system

As long as people can deal with number by matching a pebble or a notch on a stick with an object, it is not necessary to have a system of numeration. When this matching method becomes inadequate people must create a system of numeration. Our ancestors discovered a well-defined way of numbering, namely by tallying or matching. These kind of methods we call primitive systems of numeration. It is a way of checking if there is a one-to-one correspondence between two sets.

The systems that were developed after the primitive systems were the additive systems. These systems had a base but no place value. These systems of numeration usually had a base of five, ten or twenty. The number of fingers on a hand, on both hands, or the number of fingers and toes influenced the determination of the base.

The last and most efficient systems are the so-called positional systems. A positional system has a base and above all place value. Our system, the Hindu Arabic system is an example of a positional system with base ten, ten digits (0....9) and two values for each digit: a positional value and a total value.

The experimental teaching program

The teaching program follows roughly the historical line of development of our numeration system. That is, first graders are presented first of all with a primitive system , secondly with an additive system and thirdly with our positional system. All this is organised in a play which happens to take place in an animal forest. The play in the animal forest consists of three acts. In the first act Eelco the Squirrel (Eekhoorn) plays the leading part. Eelco counts and represents the result like precivilised man probabely used to do. When Eelco is gathering nuts for his supply for the winter he breaks his paw. The squirrels of the forest decide to have a meeting to discuss how to help Eelco. Doctor Squirrel suggests that some of them should take Eelco a few nuts every day. Because squirrels do not have a numeration system, Doctor Squirrel will put down a few branches next to Eelco's bed. Each day a squirrel will put down as many nuts as there are branches.

The leading player of the second act, Piet the Post-pigeon, happens to be in the shop when Eelco is making a transaction with Florus the Goblin to get him to keep charge of his nut supply. Piet the Post-pigeon is very surprised to see Eelco's roundabout way of dealing with his supply.

Piet counts according to an additive trading system with base six and two symbols: a leaf (φ) and a grass blade (/). These symbols are chosen so that Piet is able to keep quite a large number of them in his bill. Thus, Piet uses a system in which quantities are counted with two counting units. A quantity is first of all treated with the biggest unit. The number of times this takes place is represented by the leaves. The remaining number is counted with the procedure: for one blade there has to be one object. Piet represents the counted numbers by the following numerals:

/,//,///,////,/////,φ,φ/,φ//,φ///,φ////,φ/////,φφ,....,φφφφφφ/,....

Piet's system is an additive system. It uses no order or place value. We do not , therefore, have to represent the numbers in the order of symbols used above.

In the third act, Ineke and Tineke the apes appear on stage. The symbols this time are not leaves and blades but the apes's fingers. Piet suggests that one of the apes, Ineke, counts the smallest unit, i.e. the ones, and the other ape, Tineke, counts the biggest unit, i.e. the tens. As soon as a unit is counted (full) a finger is put up. A label is put on each ape, so that nobody gets mixed up. For everybody, and especially for the apes themselves, it is clear who is counting what.

The pupils are still using an additive system with base ten. There is still no order of Ineke and Tineke nor place value. As long as the labels are used no place value is necessary. It is shown, that when no labels are used, place value becomes necessary.

METHOD

Subjects

Four first- and second grade classes from four schools in two towns in the Netherlands took part in the study. The schools represented a reasonable cross-section of city and rural school populations. The particular schools used were chosen because of their willingness, especially among the teachers of the first grade, to be involved in the study.

Procedure

The children of the four classes were given a pretest when entering the first grade. The pretest was designed to assess the general cognitive-developmental level of the children before they received any systematic teaching in reading and arithmetic. The items of the pretest were of a kind ordinarily found in any sort of intelligence test for 6-year olds.

There were two experimental classes - receiving the experimental teaching program on numeration systems - and two control classes - receiving a regular teaching program - in each town.

The experimental teaching program started in January and took about two months, that is 40 lessons of half an hour. The teachers for the two experimental classes were the class teacher.

Following these lessons at the end of the school-year, posttests were administered to all four classes, these consisted of two parts: numeration items and addition- and subtraction items up to 20. Both parts were given as class-administered tests.

The battery of numeration items consisted of the following kind of items: (a) what number comes after a given number, (b) what number comes before a given number, and (c) circle the largest number of two given numbers.

Following the post-test at the end of the first grade, a very similar post-test was administered to the same four classes at the end of the second grade. The main difference between the two post-tests was that with the numeration items, three- and four digit numbers were used as well as two-digit numbers; the addition and subtraction items went up to 100.

With 62 first graders post-test items were given as individual tests by a member of the project staff. The pre- and post-tests were administered by the class teacher but always with a project-staff member present.

RESULTS

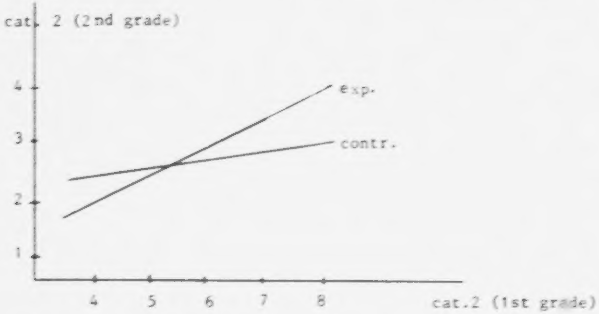
Results at the end of the first grade

The results at the end of the first grade deal with possible differences between the experimental and control group a few months after the experimental teaching program on numeration systems was given.

Because we wanted to talk with the children individually about the items, we matched pairs of pupils (one from the control group and one from the experimental group) on the basis of the pretest score. This procedure resulted in 31 pairs, that is n=31 for experimental and control group. A t-test was carried out on the data and resulted in a significant difference between the experimental and control group on numeration items and no significant difference on sum items. One would of course expect an advantage for the experimental group on numeration items. This, however, does not result in a better result on sum items. With these quite simple items one does not have to use the algorithm to get a correct answer quickly. And that is exactly what happened. In the individual sessions we noticed that 23% of the pupils in the control group used the decomposition strategy with category 2 items and the rest (76%) used all sorts of counting strategies. With the category 2 items (regrouping items), it is more efficient to use the properties of our numeration system, that is to use the decomposition strategy. It is in the experimental group that this happens: 95% used the decomposition strategy with the category 2 items and the rest a counting-on or counting-backwards strategy.

Results at the end of the second grade

To test the existence of a difference between the experimental and control group at the end of grade two, an analysis of covariance was performed on the data with the first grade category 2 items as covariate. This analysis shows no significant differences on numeration items except for the number after items. For the sum items an analysis of covariance is not allowed, which means that there is an interaction between the variables. The interaction between the category 2 items and the covariate (first grade category 2 items) is depicted in the following figure.



The regression coefficient for the experimental group is .67, which is significant (p=.001); for the control group this coefficient is .18, which is not significant (p=.175). The correlation between category 2 items in first and second grade is therefore significant for the experimental group and not for the control group. This also means that if a pupil in the experimental group scores high on the first grade category 2 items he will score high on the second grade category 2 items. There is no such relation for pupils in the control group. It does not matter what the result of a pupil in the control group is on the first grade category 2 items, his result on the second grade category 2 items will always be between 2 and 3 correct. The same kind of interaction is found for the second grade category 1 items.

The interaction is depicted for the category 2 items because for solving these kind of items the properties of our ten based positional system have to be used. As we have seen the pupils in the experimental group used the decomposition strategy mostly to solve the category 2 items in the first grade post-test. So, it could be hypothesised that there is a relation between knowing the properties of our numeration system and solving second grade category 2 items.

The control group which did not follow the experimental teaching program in the first grade did of course learn the properties of the place value system. Most of the pupils learn it eventually together with the decomposition algorithm.

The question asked in the introduction of this article however, was: does it make a difference if the place value system is taught before or together with the algorithm? This question can be answered by performing an analysis of covariance on the second grade category 2 items with the second grade numeration items as covariate.

In this analysis the b's are homogeneous: the F-value for the common b is 5.10 (p=.027), which means that there is a significant correlation between second grade category 2 items and second grade numeration items for both groups. The analysis of covariance gives a p-value of .002, which means that there is a significant difference between experimental and control group, where the experimental group is the most effective.

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2. SPATIAL REPRESENTATION AND GEOMETRICAL UNDERSTANDING

Adolescent Girls' and Boys' Ability to Communicate a Description of a 3-Dimensional Building

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ABSTRACT. This study presents evidence which suggests that students in grades 6 through 8, boys and girls, have difficulties in representing and communicating information on a 3-dimensional building, made up of cubes. Types of representations used by the students were classified and analyzed by grade (6,7,8), sex (boys, girls), site (inner city, rural, urban) and time (pre-, post-instruction). The findings were examined relative to their psychological aspects and practical teaching implications.

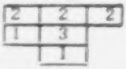
BACKGROUND. The need and reasons for emphasizing various types of representations of spatial shapes and relations are discussed in detail by Gaulin (1985). The ability to represent and interpret 3-dimensional geometrical relations is a valuable skill for many school subjects and technical occupations. Thus, providing all pupils with opportunities to explore a variety of types of representations of spatial and geometrical information, as well as to communicate such representations should be basic educational objectives. This study presents evidence which suggests that middle school students, boys and girls, have difficulties in representing and communicating information on a 3-dimensional building, made up of cubes.

PURPOSE OF THE STUDY. During a pilot study for developing spatial visualization activities appropriate for middle school students, it was found that students have difficulty in visualizing hidden parts of pictorially presented objects and in describing and representing 3-dimensional objects. In order to evaluate the effectiveness of the instructional activities, especially the practical ones dealing with representation schemes and drawings, "a building description task" was created. There were three interrelated purposes in presenting this type of task. The first was to study types of representations used by middle-school students in attempting to perform the task. The second was to investigate site, grade and sex-differences in kinds of representations used by them, and to what extent they were successful. The third purpose was to determine whether students' performance would be affected by instruction in spatial visualization activities. Several other aspects of the effectiveness of the instructional activities were measured by a spatial visualization multiple-choice test. They have been reported elsewhere (Ben-Haim, 1983; Ben-Haim et al., 1985).

THE BUILDING DESCRIPTION TASK. Every participant received a building of cubes on a plan card with the following task:

> You are seated on one side of a screen and your friend is seated on the other.
> Your friend cannot hear what you say, but you may pass a piece of paper to him.
> Your friend has a supply of cubes to work with.
> Here is a building made of cubes. You are the only person that can see the building.
> Your task is to help your friend to know what your building looks like.
> Be as creative as you wish.

Figure 1 presents the "map plan" of the building description task (the numbers in the squares indicate the height of the stacks of the cubes).



2	2	2
1	3	
	1	

Figure 1

According to Wattanawaha's DIPT (Dimension, Internalization, Presentation, Thought Process) classification system for non-speeded spatial tasks (Clements, 1983) the "building description task" falls into the highest values in each of the four independent characteristics. It is identified as D-3, I-2, P-2 and T-1.

THE METHOD OF THE STUDY. Before and after exposure to the spatial visualization activities, two to four students from each of participating 6th, 7th and 8th grade classes were randomly selected to be given the "building description task". The students in those classes were from several schools (sites) representing a broad range of socio-economic status (SES). Table 1 shows the distribution of the entire sample by time (pre-, post-instruction), grade level (6th, 7th, 8th) and sex. The unit of instruction required students to create "buildings" from small cubes and to draw representations of these "buildings" in two ways, flat front or side views, and isometric corner views. The spatial visualization unit called for 12-15 hours of instruction. None of the activities set for the students during the instruction, specifically involved tasks similar to the "building description task".

RESULTS. Evidence obtained from the students' attempts to perform the task prior to the instruction indicated usage of a variety of types of representations. These incluced verbal descriptions, graphic drawings of side views or perspective drawings, descriptions by layers, coded orthogonal views and mixed strategies. However, it was decided to follow Richardson's classification of subjects according to three representational modes identified as Verbal (V), grahpic (G) and mixed (M) which combines verbal and grahpic representations. The three modes may not be distinctive but may be points on a visual/non-visual continuum of a type suggested by Richardson (1977). Table 2 presents the percentages of students by modes of representations, grade (6th, 7th, 8th), time (pre-, post-instruction) and sex. Taking into account only the cases in which the performance was sufficiently adequate to construct the given building according to the students' descriptions, the results show a success rate of 23% prior to instruction compared to 81% after instruction with almost no differences by grade or sex and with some advantage to the mixed mode. The verbal mode was successful only for one 7th grade girl and one 8th grade boy. The analysis of 6th garde data prior to the instruction shows site differences regarding the modes of representations. The rural students used verbal and graphic modes equally, the inner city

Table 1: Distribution of entire sample by time (pre-, post-instruction), grade, and sex

		Grade			Total
		6th	7th	8th	
No. of Classes					
	Pre-	7	6	4	17
	Post-	3	7	5	15
No. of Subjects					
	Pre-	29	19	14	62
	Boys	15	13	7	35
	Girls	14	6	7	27
	Post-	11	26	15	52
	Boys	7	14	8	29
	Girls	4	12	7	23

Table 2: Percentages of students by modes of representations (Verbal, Graphic, Mixed), by grade, time (pre-, post-instruction) and sex

		Pre-			Post-	
	V	G	M	V	G	M
Grade 6	17	48	35	0	27	73
boys	13	47	40	0	43	57
girls	21	50	29	0	0	100
Grade 7	53	37	10	4	73	23
boys	46	46	8	0	100	0
girls	66	17	17	8	42	50
Grade 8	43	7	50	0	73	23
boys	43	14	43	0	75	25
girls	43	0	57	0	71	29
Total	34	35	31	2	63	35
boys	31	40	29	0	79	21
girls	37	30	33	4	44	52

students used the graphic and mixed modes more frequently and the urban students used only the mixed mode and were most successful in their attempts to perform the task.

CONCLUSIONS. Prior to instructional intervention, the findings of this study suggest that:

(i) a great variety of types of representations are used by middle school students to respond to an identical spatial task with minor grade and site differences but with no sex differences.

(ii) only about 25% of the 6th through 8th grade students can perform successfully on a task such as the "building description task" with no grade or sex differences.

However, after instructional intervention in spatial visualization activities, the results demonstrate:

(iii) a shift from the verbal representation mode to grahpic and mixed modes in all three grade levels, with girls moving more towards the mixed mode and boys towards the grahpic mode.

(iv) a significant improvement in performance on the building description task (from 23% to 81%) regardless of grade level and sex.

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REPRESENTATIONS OF THREE-DIMENSIONAL FIGURES BY MATHEMATICS TEACHERS-IN-TRAINING

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Background

In the classroom, two-dimensional drawings are frequently used to 'clarify' some concept or information related to a three-dimensional frame of reference, it being assumed that the learners can interpret the graphical material which is presented. Examples of this abound in mathematics textbooks in common use. A comparison of the content of older and more recently published mathematics texts led Polack to conclude: "In the textbooks examined, students were asked...to make use of more and more pictorial and diagrammatic material" (1985:16). Ben Haim et al (1985) demonstrated that 10-13 year olds often have difficulty relating quasi-perspective drawings to the rectangular solids they represent. Sex differences in linking three-dimensional shapes to their two-dimensional representation have also been reported(Connor & Serbin, 1985: Chipman & Wilson 1985). Nethertheless, learners themselves often represent a three-dimensional structure by means of some form of two-dimensional drawing. We have labelled these two modes exhibited by learners 'interpretation' and 'representation' respectively.

Gaulin (1985) invited 10-18 year old pupils in Quebec to describe simple solids made up of small cubes glued together. The types spontaneously given by subjects were categorized into a number of classes, including verbal descriptions, side views, descriptions by layers, 'coded orthogonal views' and attempts at perspective drawing. Apart from the first category, all the above classes involve two-dimensional drawings of some sort. It appears then, that children can and do use a variety of different types of paper and pencil representation to describe three-dimensional structure. Howver, teachers seldom use representations outside quasi-perspective drawings of the 'parallel projection' or isometric types (see Fig. 1).

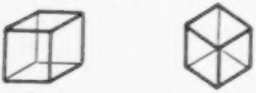

Figure 1

In fact, it is often easy to "recognise the section on space geometry in many textbooks by stereotyped figures ..." (Goddijn and Kindt, 1985).

Description of study

Purpose: 1. To examine the distribution and range of verbal and graphical representations of three-dimensional figures produced by mathematics teacher-trainees and the interpretations which they make of such representations.

2. To analyse the descriptions of three-dimensional solids produced by mathematics teacher-trainees using the Krutetskii (1976) distinction between 'geometers' and 'analytics'. (Geometers "feel a need to interpret visually an expression of an abstract mathematical relationship" (1976:321), whereas analytics "have no need for visual supports for visualizing objects or patterns in problem solving, even when the mathematical relations given in the problem 'suggest' visual concepts." (1976:317)

Proceedure: Subjects were drawn from three mathematics teacher education courses in London (N=62), Sydney (N=21) and Melbourne (N=48). Approximately two-thirds, (88) were females. In each course, subjects were divided into two groups which were examined out of sight of each other. Two of the solids used by Gaulin were employed in the study; these are represented isometrically in Figure 2.

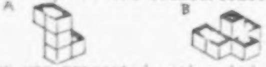

Figure 2

Each subject in the first group was presented with solid 'A', those in the second group being given solid 'B'. Subjects were then given the following instructions:

> You have in front of you a shape which you have seen children use in the classroom. You would like to buy it for use during your next practice-teaching experience, but do not know from which distributor it is avail-

able. Write a description of the shape, which you could send to a number of distributors. You may use words and/or diagrams.

When subjects had completed their descriptions, the solids were removed and a written description of the other solid was provided together with a heap of small cubes. Subjects were given the actual solid described and asked to comment on the efficacy of the description from which they worked.

Results

Representation

In producing descriptions of the solids presented to them, subjects articulated a variety of responses. These included plans and elevations which were usually labelled or coded, quasi-perspective drawings, and unillustrated verbal descriptions. Many of the graphical responses were accompanied by verbal material which ranged from scant notes to near-complete descriptions. A few subjects gave independent graphical and verbal responses which were complete in their own right.

In Table 1, a summary of the broad categories of response for the respective solids is presented. A drawing "with few words" describes a response in which the verbal part consists of labels or very scant notes. The phrase "with many words", on the other hand, indicates a response in which the drawing(s) would serve as an adequate description without any more of the accompanying commentary than a few words used essentially as labels. The term "illustrated verbal description" is used for a response in which a near-complete verbal description refers to graphical material which accompanies it but which would not suffice without it.

	drawing only	drawing with few words	drawing with many words	illustrated verbal description	words only
Successful Outcome					
block A	2	11	48	5	6
block B	2	12	31	18	2
Unsuccessful Outcome					
block A	2	0	2	1	1
block B	1	1	1	2	2

Table 1 Frequencies of response types

As can be observed, a very small proportion of subjects were soley "analytic" or "geometric", in Krutetskii's terms. On the other hand, if responses in the first two columns are taken as being those of geometric types and those in the last three as representing analytic types, it can be seen that the majority of these mathematics teachers-in-training can be typed as analytics. These are individuals who rely heavily on verbal descriptions. Inspection of the results separately for each sex revealed no appreciable differences.

The graphical representations varied considerably. Many were quasi-perspective (quasi-isometric) drawings. Others were plans (views"from the top"), which were usually coded or annoted. Some of the illustrations involved elevations such as "front view" or "side view", which were occasionally coded. The different responses are adquately described by Gaulin's (1985) categories of side views, description by layers, coded orthogonal views, and attempts at perspective drawings (quasi-perspective). The distribution of graphical responses across the various types mentioned above is given for each kind of block in Table 2.

	Type of graphical representation			
	side view	description by layer	coded or orthogonal view	quasi-perspective
Successful Outcome				
block A	5	4	7	51
block B	8	6	28	31
Unsuccessful Outcome				
block A	2			3
block B	1		1	2

Table 2 Frequencies for response for each type of drawing

Note: Because some respondents used multiple graphical representations the entries in the Table exceed the number in the sample.

It may be observed that quasi-perspective drawings were used more often for the simpler structure (block A) than for the more complicated one, and that this type of representation was by far the most popular overall.

Interpretation Most of the subjects correctly interpreted the verbal descriptions they were given, although due to a failure in communication between the three researchers, the method used by two were not the same as that used by the third. Two of the researchers asked the subjects to comment on the efficacy of descriptions written by other subjects whereas the third offered the same description to all subjects. In this latter case, the only mistake was the production of mirror image of the structure actually described. In the former case, when both verbal and graphical descriptions were available, subjects typically found the latter more helful, as can be seen from comments like: "Diagram most useful. Description less so". A number of other subjects commented that they appreciated being able to refer to a verbal description to 'check' their interpretation of the diagram.

Conclusions

Despite the age difference and the mathematically more sophisticated nature of some of our sample, the findings are consistent with those of Gaulin. In particular, these prospective teachers were mostly able to describe the three-dimensional solids sufficiently clearly for others to construct the shape from the description provided. They nearly all did so using words as well as drawings although, overwhelmingly, they considered the drawings to be more useful than words when they came to use the description. The most popularly descriptive form was the quasi-perspective drawing, particularly in the case of the simpler block (Block A). For the more complex structure, graphical representation was equally divided between quasi-perspective drawings and simplified two-dimensional representations. No significant sex differences in interpretation or representation of the graphical material or three-dimensional solids respectively were evident in this sample.

The category which would repay further study are those classified as 'unsuccessful' outcomes'. These were the subjects in two of the participating institutions who were unable to construct the three-dimensional block from the description provided from another subject. In one institution this comprised 10 out of 62 respondents, equally divided between men and women and equally divided between the two blocks, A and B. It would be useful to look at a larger sample in order to see if the failure rate was consistant.

Inability to represent a three-dimensional object either verbally or diagrammatically must have important implications for the future practice of these teachers. The overwhelming preference for a diagrammatic representation which conforms with usual educational practice only reinforces the suspicion that formality of presentation is validation within the system and lack of formal technique could be tied closely to feelings of mathematical incompetence.

Finally, it would be interesting to pursue the relationship between teachers' preferred diagrammatic representation and those of their pupils in order to establish whether a discontinuity is another explanation for misunderstanding or lack of comprehension is geometry.

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THE ACQUISITION OF SOME INTUITIVE GEOMETRICAL NOTIONS IN THE AGES OF 3-7: COGNITIVE GAINS ACQUIRED THROUGH THE AGAM METHOD

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Many of the difficulties that students encounter in the study of geometry can be attributed to: (a) the lack of an intuitive basis of geometrical concepts - levels I and II of development in geometrical thinking as defined by the van Hieles (1958); (b) deficiencies in visual skills. Quite often, students complain that even when they know what they are looking for, say in a process of designing a proof, they cannot "find" it in the "forest" of lines. Even simple recognition of basic geometrical shapes (e.g. a right-angled triangle) can become a difficult task for students and even for their teachers when these shapes are rotated (Hershkowitz & Vinner, 1984).

Attainment of the various levels as defined by the van Hieles is not regarded as a spontaneous process dependent only on biological growth and on age, but also on content and methods of instruction (Wirzup, 1976). But within the regular school curriculum no systematic effort is made to develop visual abilities. Common practice limits visual instruction to art education, though many researchers and educators agree that the processes involved are of general significance, as well as specific importance for geometrical, mathematical and scientific thinking, and should therefore be trained as other basic skills are, e.g., verbal skills (Freeman, 1980). In accordance with these considerations, Yaacov Agam (Agam, van Dalen Garcia, Gattone de Rivera, 1981) designed a general basic program in visual education for children in the preschool and lower grades of primary school.

The Science Teaching Department at the Weizmann Institute of Science has been involved, since 1983, in implementing and studying the Agam Program. The program consists of 36 units that are listed in Table 1. These units constitute a course lasting 2 to 4 years for children between the ages of 3 and 7 years. It teaches systematically basic geometrical concepts and their relationships, as well as visual skills. The program employs a rather unique didactic approach which allows children to acquire sound intuitive geometrical notions at a young age. It can, therefore, play an important role in the development of the lower levels of geometrical thinking.

Table 1

The units in the Agam Program

1. Circle
2. Square
3. Patterns
4. Circle & Square
5. Flash Identification
6. Horizontal
7. Vertical
8. Horizontal & Vertical
9. Oblique
10. Horizontal, Vertical & Oblique
11. Triangle
12. Circle, Square & Triangle
13. Variations of Forms
14. Symmetry
15. Curved Line
16. Large, Medium & Small
17. Angles
18. Point
19. Typical Forms
20. Proportions
21. Red
22. Yellow
23. Blue
24. Secondary Colors
25. White, Black & Gray
26. Trajectory
27. From Eye to Hand
28. Numerical Intuition
29. Composition
30. First Dimension
31. Second Dimension
32. Third Dimension
33. Fourth Dimension
34. Letters
35. Visual Grammar
36. Creativity

Method

During the school years 1983/84 - 1984/85 the program was implemented in 4 preschools in Israel and was subjected to extensive research. An experimental (4 preschools, 83 children) versus comparison group (4 preschools, 49 children) design was used. The four comparison preschools received no training and were matched pairwise with the four experimental preschools. Tests were created to measure the effects of the Agam Program on visual identification, memorization and reproduction as these skills relate to the contents of each unit. These tests were administered after each unit, or after several related units. In addition, summary tests were given at the end of the first year and at the end of the second year. About half of the items in these tests were based on activities similar to those in which the children received training. The other half consisted of items that tested different measures of transfer.

Results and Discussion

We describe results on a selected set of items illustrating some cognitive benefits, relevant to mathematics education, that children gain through the Agam Program.

(a) *Differentiation of concepts.* The results indicated that both the experimental and the comparison children apparently developed naturally, i.e., without the special instruction of the Agam Program, many geometrical concepts. In the experimental group, however, these concepts served as a basis for a directed development of more refined concepts. One example is the concept of a periodic series of geometrical shapes. After a brief introduction of the concept to the comparison group, about 90% of the children were able to provide an example of the concept. A similar proportion was able to do so in the experimental group after it had been given instruction in the relevant unit, *Patterns*, indicating an equality between the groups on a basic level of cognitive development. However, when the same groups were given a debugging task, considerable differences emerged. The children were given four different series of geometrical shapes. They were told that some of these series were periodic while others contained an error and needed to be corrected. For example, one of the bugged patterns was the following: □□○□□○□○□□ On the average, 71% of the children in the experimental group (N=55) succeeded in this task vs. 44% (n=16) in the comparison group ($p<.0004$, based on an analysis of covariance with IQ serving as a covariate). In another task the children were asked to draw the continuation of four different periodic series. The average score on the four tasks was 86% in the experimental group vs. 68% in the comparison group ($p<0.03$). These results suggest that the Agam Program brought about differences on a higher level of concept formation. Higher levels of concept differentiation in the experimental group were found also in other units of the Agam Program. Examples involve distinction between square and quadrilateral, circle and closed curved line (e.g. ellipse), identical and symmetrical shapes, etc.

(b) *Sensitivity to instances of geometrical concepts in one's environment.* At the end of each unit, children were asked individually to identify in their classroom instances embedding the concept studied in that unit. Figure 1 presents results for the first four units. The overall difference is statistically significant with $p<0.00005$. There were no differences between the groups in the number of incorrectly identified instances.

Figure 1

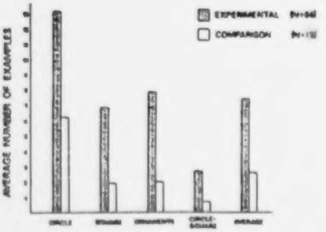

Figure 1. Average Number of Correct Examples (Corrected for Intelligence) Given by Children in the Experimental and Comparison Groups in Each Unit.

These results seem to indicate that the program "opens" the children's eyes to geometrical features of their environment. It is assumed by the Agam Program that the increased visual sensitivity and ability to analyze geometrical features in the environment will generalize and transfer, so that the trained child will have a general advantage in identifying a certain geometrical feature when needed. Other results indicated, however, that the effect of the program was not restricted to an improvement in the children's skill of looking at their environment analytically. They were also better than the comparison group at global perception when this was required. One example comes from data on the children's performance in memory tasks.

(c) *Memory Development.* In every test children were shown briefly one or more "memory cards". Subsequently, they had to identify the same design among nine drawings in their test booklets. The results indicated consistent superiority of the experimental group. For instance in the unit *patterns*, the average success rate in the experimental group was 41% vs. 22% in the comparison group ($p<0.005$).

Another result which suggests better utilization of short term memory was the higher complexity of the periodic series drawn by children in the experimental group. The average number of elements per cycle of the series was 2.4 in the experimental group vs. 1.6 elements in the comparison group ($p<0.003$). In addition, the average number of shapes in one element of the series was 1.9 in the experimental group vs. 1.6 in the comparison group ($p<0.002$). Some children as young as 3 1/2 years old trained in the program drew amazingly complex periodic series. Analysis of the information that has to be kept in mind by the child to produce such a series, highlights the great effect that mastery of the concept has on reasoning.

(d) *Reproduction from memory.* A related finding is illustrated in the reproduction tasks that were given to the children. In one of these tasks, children were shown briefly a combination of geometrical shapes such as the following: . The children subsequently had to reproduce the shape combination from memory using transparencies on which squares of different sizes were printed, one square to a transparency. There were no differences between the groups in correct identification of the elements from which the shape combination was composed, indicating no difference between the groups in the analytic part of this task. But there were considerable differences in favor of the experimental children in the ability to reproduce the spatial relationships between the elements, pointing to an experimental effect on that part of the task that requires synthesis of the disparate elements into a whole.

The summary tests given at the end of each year examined additional issues including: span of visual short-term memory, visual flexibility (e.g. identification of the various triangles that are embedded in a Shield-of-David type stimulus, a task testing the ability to see the same lines as part of different shapes), mental rotation of shapes, visual sensitivity and accuracy, spatial orientation, verbal vs. visual encoding, motor skills and field dependence. Results from these tasks relevant to geometry will be discussed in the presentation. A more detailed description of the study is given in Eylon et al. (1985).

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SIMILARITY CONCEPTS AT THE GRADE LEVELS 6, 7, 8.

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INTRODUCTION

Phenomena that require familiarity with enlargement, scale factor, Projection area growth, and indirect measurement are frequently encountered by children in their surroundings. To understand these phenomena children must develop a concept of similarity as Part of the overall geometrical understanding of their environment. Because of its visual representation, similarity may also be a first step towards an understanding of proportional reasoning.

The present study has two major concerns: (1) finding patterns in children's development of the concepts of similarity and area growth over grade levels 6-8, and (2) describing the effects of a specific instructional intervention over these three grade levels.

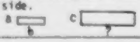
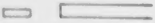
METHODOLOGY

The intervention consists of a two-three week-long instruction with the Middle Grades Mathematics Project (MGMP) Similarity Unit. The unit provides a carefully sequenced set of challenging and exploratory activities designed to fit the Van Hiele levels I (properties of shapes), and II (relationships among properties of shapes), attempting thus to build a base for further advance in the understanding of geometrical concepts in general, and of similarity concepts in particular.

Fifty average-ability students from two classes at each of the grade levels six, seven, eight comprise the sample for the study. These students underwent both before, and after instruction a 30-60 minute-long interview.

The interview contained five tasks - four on rectangle similarity and one on the area relationship of similar shapes. The first four tasks were varied to test the student's facility with handling proportions of increasing numerical difficulty (see Table 1), whereas the Area Task was presented in three increasing levels of concreteness.

Table 1: Interview questions

NUMERICAL TYPE a by b c by d TASK	1 $a \mid b$ $a \mid c$	2 $a \nmid b$ $a \mid c$	3 $a \mid b$ $a \nmid c$	4 $a \nmid b$ $a \nmid c$
1. Decide whether two drawn rectangles are similar or not. 	3 by 6 and 9 by 18	2 by 3 and 8 by 12	3 by 9 and 4 by 12	6 by 8 and 9 by 12
2. Decide whether two cut-out rectangles are similar or not 	2 by 4 and 6 by 12	2 by 3 and 6 by 9	2 by 6 and 3 by 9	4 by 10 and 6 by 15
3. Given the lengths of three sides of two similar rectangles, find the fourth side. 	2 by 6 and 6 by ?	2 by 5 and 6 by ?	4 by 12 and 7 by ?	6 by 10 and 9 by ?
4. Cut a strip to make a rectangle similar to a given one. 	2 by 4 and 6 by ?	2 by 3 and 8 by ?	2 by 6 and 5 by ?	4 by 6 and 6 by ?

In the latter task, students were shown a rectangle representing a small room which cost $300 to carpet. The interviewer asked the student what would be the price of carpeting a larger room which is three times as long and three times as wide. The task was presented first at the least level of concreteness (no additional illustration), and the student was led through level 2 (choosing the representation of the larger room from an assortment of given cut-out shapes), and level 3 ("building" the larger room from several given cut-out copies of the smaller room). If the student gave a correct answer at some level, the interviewer did not proceed further.

MAIN RESULTS

Rectangle Similarity. The level of success of the interviewed students on the rectangle similarity tasks increased with grade level and depended also on the numbers (Numerical Types) involved. A comparison of the average number of correct answers pre/post instruction shows a considerable improvement in performance on each of the four tasks and numerical types for all grade levels.

A strong influence of the numerical type may also be observed: for all tasks and grade levels, there is a considerable gap (20-45 percent) between level of performance with numbers that are divisible across rectangles (Types 1 and 2) and between cases in which such comparisons do not render whole numbers (Types 3 and 4). It should also be mentioned that after instruction, an average mastery level of above 80 percent has been achieved for the first two numerical types but not for the others.

Student strategies in the rectangle similarity tasks followed roughly the classification of responses for proportionality tasks indicated in Karplus & Karplus (1972): visualization, addition, multiplication and adjustment, whole multiplication, and proportional reasoning.

Table 2 presents a summary of the distribution of strategies employed in Tasks 1 and 4 for Numerical Types 3 and 4 by the whole sample before and after instruction. In the two matrices, the numbers located on the main diagonal indicate no change of strategy between the two interviews. The upper and the lower halves, with respect to the diagonal, mark students that employed respectively more advanced or lower strategies in the post-interview as compared to the strategy employed in the same task in the pre-interview. The results for both Tasks 1 and 4 indicate that about 90 percent of the students were either stable or advanced (with an almost equal division of 45 percent for each of the two categories) and only about 10 percent employed in the post-interview lower level strategies than they did in the pre-interviews.

Table 2:

Distribution (in percent) of student strategies employed pre/post instruction on Tasks 1 and 4 for Numerical Types 3 and 4.

POST PRE	1	2	3	4	5
1	3			11	12
2			1	1	4
3					1
4	3	4		27	14
5	1	1		2	15

Task 1

POST PRE	1	2	3	4	5
1	4	3	1	6	12
2		5	1	2	10
3					2
4		6		16	9
5			2	1	20

Task 4

Area Task. The following strategies could be detected on the Area Task: no reasoning, linear scaling, scaling and adjustment, other scaling, peripheral counting, and area scaling.

The Area Task strategies mentioned above are hierarchically arranged according to arguments presented by Lunzer (1973). According to him, the use of the linear scale indicates an ability to operate at a concrete operational level, whereas the dissociation of the area growth from the linear growth requires formal thinking. Therefore, in this hierarchy, the linear scaling strategy precedes any other multiplicative strategy. The other strategies were ordered according to what was considered by these investigators an increasing degree of insight into the relationship at hand.

As in the case of the Rectangle Similarity Tasks, student progress may be measured by the percent of students that employed at the post-interview a more advanced strategy as compared to their pre-instructional performance: 42 percent of the students were found to belong to this category, 38 percent used the same strategy on both interviews (this includes those students using a correct strategy pre and post), and 18 percent used a lower-level strategy after instruction.

The findings also indicate that the level of performance on the Area Task was particularly low for the sixth graders and no improvement could be detected for these students. The findings are consistent with a study by Fitzgerald and Shroyer (1979) on the effect of instruction in area and volume growth at the sixth grade level. On the other hand, the seventh and eighth graders showed a better performance at the initial stage than sixth graders, and also significant gains as a result of instruction. An analysis of strategies revealed that after the instructional intervention above 80 percent of the interviewed students employed the same, or a more advanced strategy at the first attempt (i.e., at the least concrete presentation) of the Area Task.

CONCLUSIONS

From a pedagogical point of view, teaching similarity at grade levels 6-8 seems to be a rewarding experience - particularly for the last two grade levels. The study shows that children's understanding of geometrical similarity may be improved through instruction. A careful selection of the numbers involved in proportions is obviously needed in both teaching and testing any phenomenon that requires proportional reasoning.

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LOGO AND THE NOTION OF ANGLE AMONG FOURTH AND SIXTH GRADE CHILDREN

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There has never been a single definition of angle which has been widely accepted. As Close (1982) points out, the earliest definitions considered angle as a form of distance. Modern definitions tend to fall into one of two categories, static (eg, an angle is a part of the plane included between two rays meeting in a point) or dynamic (eg, an angle is the amount of rotation necessary to bring one of its rays to the other ray without moving out of the plane). However, many texts which begin initially with static angle definitions eventually refer to angles as rotations. According to Heath (1956), this indicates the essential nature of angles -- that of rotation.

The right angle was the first commonly used unit of angular measure, and was applied mainly to static angles. The use of the degree evolved later. It was considered to be a measure of both the length of an arc and of the angle at the centre subtending it. Measuring an angle in degrees was based on a dynamic notion, that of the assumed circular rotation of the planets.

In the teaching of school geometry, Freudenthal (1973) stresses the need for both static and dynamic definitions of angle. He also suggests introducing measurement of angles at the same time as angle concepts, as a means of developing understanding of the latter.

Logo, a computer programming environment appears to be an ideal vehicle for experiencing a dynamic approach to angles and to their measurement. The rotation of the Logo turtle can be considered an example of a dynamic angle; the input to the turtle turn, as a measurement of that angle. The Logo research studies described

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[°] I am grateful for the collaboration of Alain Taurisson in the 4th grade study and of Joel Hillel and Stanley Erlwanger in the 6th grade study.

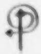
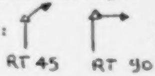
in this paper had as one of their aims the investigation of the evolution of children's concepts of angle and of angle relationships.

FOURTH GRADE STUDY

Method. This study, carried out during the school year 1984-1985 involved 19 10- and 11-year-olds, 14 from a traditional school and 5 from a non-traditional school. In the former, one computer occupied a corner of the classroom. Children worked at the computer alone or in pairs while the teacher and the rest of the class worked on non-Logo activities. In the latter environment, the children had access to computers in a lab. They worked in pairs for an hour each week. There was considerable adult help available whenever they wished. The children in both groups experimented with graphics projects of their own choosing. The study also included a small control group of 5 children from the traditional school. They were in another fourth grade classroom which did not have access to a computer.

Three individual interviews were carried out with each child throughout the year -- in September, January, and June. The same questionnaire (with slight modifications for the control group) was used for all groups of children. Follow-up interviews were carried out in March 1986 with five of the children from the raditional school who currently have a Logo-equipped computer in their grade 5 classroom.

Findings. The results are presented in three sections: A) the findings of the first two interviews; B) the results of the third interview; C) the findings of the recent follow-up interview.

A) By mid year, it became apparent that the children had a concept of angle which was static (eg, a corner, a surface, or a slanted line) and another distinct concept of rotation. Furthermore, they experienced difficulty in predicting what the turtle would do when they typed in various inputs to RT. For example, RT 120 might result in and RT 180 in  . They were not able to perceive the relationship between the input to the angle of rotation and the relative size of the constructed angle ("RT 45 is smaller than RT 90, but we end up with something bigger": 

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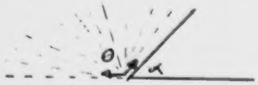
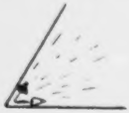
All of these difficulties led us to modify somewhat the Logo environment of the class in the traditional school. We introduced a LASER TURTLE, a turtle which illuminates the screen with laser beams every 5 degrees while it slowly turns through the angle of rotation. After the rotation is completed the beams are erased one at a time. The classroom introduction of the LASER TURTLE to the children was accompanied by examples where i) the angle of rotation θ was supplementary to the constructed angle α ($\theta = 180 - \alpha$) and ii) the angle of rotation was equal to the constructed angle ($\theta = \alpha$).

i) 

ii) 

⇧ = initial heading

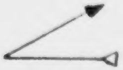
↑ = final heading

It was hoped that this modification of the Logo environment would help the children to a) see more clearly what happens when they provide a certain input to RT or LT and thereby be able to predict the final heading of the turtle and b) link together their static concept of angle and their dynamic concept of rotation.

B) The third interview held in June indicated that;

i) many children were still confused about whether the input to a turn was the size of the constructed angle or the size of the angle of rotation.

Question: "What do you have to say to the turtle to get it from the white starting position to the black final position?"



Responses: Laser group: 5/14 correctly used inputs for the turn which exceeded 100.

9/14 used inputs less than 60.

non-Laser group: 4/5 used inputs between 130 and 140.

1/5 used input of 50.

ii) the children seemed to find it easier to draw a figure which corresponds to a given command than to provide the command which is needed to draw a given figure.

Question: "Please draw what you think the turtle will do if you give it these commands:"

a) FD 45 RT 45 FD 45 (Correct: 9/14 Laser; 3/5 non-Laser. Compare with frequencies above)

b) FD 45 RT 120 FD 45 (Correct: 10/14 Laser; 3/5 non-Laser)

c) FD 45 RT 180 FD 45 (Correct: 5/14 Laser; 5/5 non-Laser).

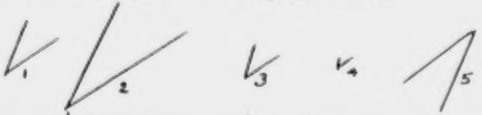
iii) most of the children continued to have a static notion of the term "angle".

Question: "What does 'angle' mean to you?"

Responses:	Laser group	non-Laser group	Control group
Line	0	2	0
Slanted side	4	0	0
Corner	5	1	4
Triangle	1	0	0
Turn in degrees	1	2	0
Two lines which meet	2	0	1
Don't know	1	0	0

iv) most children classified the size of angles according to the length of the arms.

Question: "Which angle is largest? 2nd largest? .... "



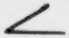
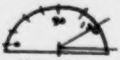
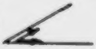
	Laser group	non-Laser group	Control group
They're all the same	5	2	1
2, 5, 1, 3, 4	8	3	4
4, 3, 1, 5, 2	1	0	0

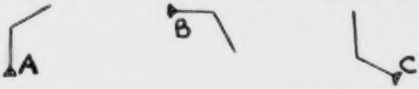
C) The follow-up interview held in March of this year with 5 of the children from last year's study showed little change in their views of angle and turtle turns, despite increased experience with Logo this year. Three children continued to view an angle as a slanted line and its measure according to the length of the line. Of the other two children, one continued to define an angle as a corner and to measure it according to "the opening" near the vertex. He also used as inputs to turtle turns the size of the constructed angle, even if it was the supplement which was required.The last child of this group was the only one who defined angle as a "turn", just as she had last June. However, there was an improvement in her ability to correctly provide the input to the turtle turn. She was now able to distinguish between the input for the angle of rotation and the input for the constructed angle.

SIXTH GRADE STUDY

Method. This study was carried out during the school year 1985-86. It involved six 12-year-olds who came to the computer lab at the university once a week. Except for the first 5 sessions when we initiated all the activities, the children worked on their own projects at least some of the time. The Logo environment had certain distinctive features, such as, the procedures Tee and Vee (state-transparent objects which the children were to use for rotation and translation activities), MOVE (left no trace on the screen), TRT and TLT (slowed-down versions of RT, LT), and absence of FD and BK for the first half of the year.

Observation notes were taken during each session. These included our interventions whenever they took place, and the children's comments, discussions, etc. These notes were used to annotate the dribble files of each pair's work. In addition, individual interviews were carried out with each child midway through the study.

__Findings__. The mid-year interview showed that the children of this study had a static view of angle. They did not, however, all share the same view as to how angles are measured. One child said that this angle :  was about 120 degrees because he visualized :  , although he claimed that the turtle would do a TLT 30 to make this turn :  . He did not perceive the equality relationship between the input to this turtle turn and the size of the angle. Another child looked at arm length to classify angles according to size. For the remaining children who were able to judge more or less the input to turtle turns and the size of static angles, there was only one who was able to coordinate the relationship between an angle of rotation and the adjacent constructed angle. For example, one task was, "If you go for a walk on these three roads, and you start at A, B, and C, on which road do you have to turn the most when you turn the corner?",



followed by (using the same diagram), "Which of these three angles is largest, 2nd largest?"

Only one child said that all the corner turns were the same and that all the angles were the same. Another said that all the turns were the same, about 45 degrees, but that angle C was largest (about 160°) angle B was next (about 135°). The other two ordered the turns one way, but did not order the angles inversely.

Throughout the second half of the study, the majority of the children continued to rely on perceptual cues in their first attempts at determining the input to turtle turns (see Kieran, Erlwanger, and Hillel, 1986, for more details). They only shifted to an analytical approach when the initial perceptual attempts failed. The fourth graders, on the other hand, did not indicate that they were able to carry out this shifting. They remained bound to perceptual cues.

Discussion. Most children are not going to acquire the powerful mathematical ideas underlying Logo without a good nudge now and then. This was evident from the Grade Four study where the children in the non-traditional school setting, who had access to adult help whenever they needed it, progressed much faster. It was also evident in the Grade Six study where we clearly had to intervene in order to draw their attention to the supplementarity relationship inherent in many Logo situations.

The question of whether Logo helps to provide children with a dynamic concept of angle is more difficult to answer. The fourth graders, after one year of Logo, seemed to keep static "angles" and their measurement in one mental compartment and dynamic turns and their input in another. The sixth graders seemed more able to integrate the two. However, as mentioned above, their first attempts at supplying the input to turtle turns were usually always based on the rationale, "it looks like...". Even when precise data were both available and known to them, they relied on perceptual cues. This was the case whether they were talking about a turtle turn or about the size of a given angle. Fourth graders also relied strongly on perceptual cues, but they weren't as good at it as were the sixth graders.

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GEOMETRICAL CONSTRUCTIONS AND THE MICROCOMPUTER

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Introduction

Solving construction problems in Euclidean Geometry requires a higher level of understanding than other topics in high school geometry because the student is expected ***to actively operate*** on concepts and not only to use them (Thompson, 1985). To know that the distance between two parallel lines is constant is one thing, and to use this in order to ***construct*** the locus of all points which are at a given distance from a line is quite a different thing, demanding a deeper understanding and mental operations.

A microworld may be defined as a closed learning environment consisting of a set of objects, a set of operations on the objects and a set of laws governing the application of the operations (Groen and Kieran, 1983). Geometric constructions may be considered a microworld in which the objects are segments and angles, the operations are the "basic constructions" and the laws are the laws of deductive Euclidean Geometry. A microworld of this type, based on a small number of operations, is appropriate for implementation on a computer and has considerable didactic potential (Dreyfus, 1984). The fact that the program does not allow the use of any operations or arguments except those that belong to the microworld is an advantage rather than a limitation. Moreover, by executing the basic operations, the computer allows the student to focus on the main issue, the analysis of the problem. Such a microworld for learning the rules of the game "Geometric Constructions" has been designed for use by the upper third of the grades 9 and 10 school population.

Stages of Solution in Construction Problems

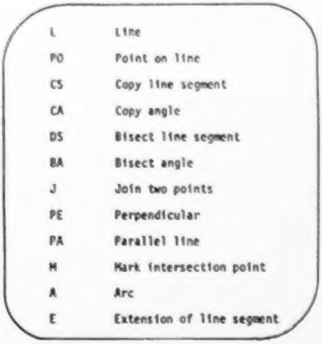
The usual procedure for solving a construction problem includes several stages, each with its own difficulties, conceptual or technical.

1. Analysis: The student has to analyze the use of data in the sequence of basic constructions that is required for the solution of the given problem. This is a deductive process which is the most important and challenging part of the solution process.
2. Selecting suitable data: After the analysis the student needs to select relevant and suitable segments and angles. Here, he has to consider some mathematical limitations: For example, a triangle can not be constructed from any three segments.
3. The construction: At this stage the student has to overcome the technical difficulties posed by being limited to straightedge and compass only. Moreover, the student needs to repeat the same basic constructions over and over.
4. Description of the construction: The student must give a written account of his construction because the end-product alone does not indicate whether the construction was made according to a correct deductive process.
5. Validity of the construction: The student must use mathematical theorems to prove the validity of his construction. In many respects, this stage is complementary to the first stage, and it exhibits all the difficulties of geometric proofs in general.

The Software

The student chooses the input data himself according to the problem which he may have received from the teacher, the textbook, the computer or which he may have devised himself. The software presents a list of basic commands, from which the student chooses the operations (the basic constructions) to carry out on his chosen data (see Figure 1).

Example: In order to construct a triangle, given a side, the altitude to this side and one of the angles formed with this side, the student will choose two segments and an angle.



L	Line
PO	Point on line
CS	Copy line segment
CA	Copy angle
DS	Bisect line segment
BA	Bisect angle
J	Join two points
PE	Perpendicular
PA	Parallel line
M	Mark intersection point
A	Arc
E	Extension of line segment

Fig. 1 The basic constructions

In the process of solving this problem, the student has to construct a perpendicular to a line q at a point D on the line. He needs to enter PE (for perpendicular) upon which the program will ask "Perpendicular to which line?" The student enters q. The program asks "At which point?" The student enters D. The entire command now appears on the screen: Construction of the perpendicular to the line q at the point D.

In this microworld the basic constructions are tools to be used in the deductive process of more complex geometric constructions. Another microworld for geometric constructions has been developed by Schwartz and Yerushalmi; the main difference between the two is that our microworld is designed so as to allow the student to build up constructions *deductively*, whereas theirs is designed to enable the student to discover geometrical theorems *inductively*.

Research hypotheses

We conjecture that we have built an immediately self-correcting and self-regulating system in the sense of Groen and Kieran (1983, p. 359, p. 372). This is achieved by the immediate feedback property which leads to improvements in the following three levels of the cognitive process involved in the solution of a geometric construction problem:

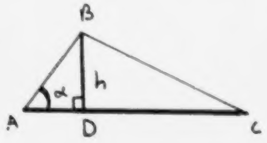
1. Conflicts that arise during the interaction with the microworld improve the student's ability to analyze geometrical problems.
2. Using the microworld, the student learns the mathematical rules of geometric constructions.
3. The formal syntactic structure of the solution process of geometric construction problems is acquired through use of the microworld.

Selected Observations

This research was based on observations of student-microworld interactions. We have used the microworld with a number of 10th graders who have learned Euclidean Geometry but had not been taught constructions. Each of them was observed while he used the computer program to solve five construction problems. Each student was first presented with a computer-guided demonstration example. The focus of the observer's attention was the students' behavior and their responses, mainly in conflict situations.

The description of the following problem solution (which was already mentioned above) illustrates some typical student behaviors: Construct a triangle ABC from the following data:

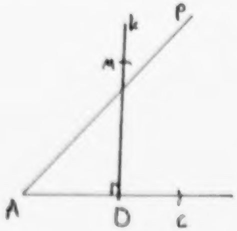
The side $b = AC$,
the altitude h on b,
the angle $\alpha = \sphericalangle BAC$.



Some of these typical observations are the following:

a) All students started the solution by typing CS, copy segment; but a segment can only be copied onto a line; when they were asked by the program, onto which line and from which point on that line the segment should be copied, they discovered their syntactic error; they cancelled the CS command and started from scratch first requesting L, a line; some were helped at this stage by drawing their attention to the list of available commands.

b) After copying the segment AC and the angle α (whose other side is called p), one of the students, Dan, chose and marked an arbitrary point D on the segment, built the perpendicular k to AC at D and asked the program to copy the given altitude h onto k from the point D. At this moment, the observer asked him where he expected the other end of h to be; he pointed to the intersection of k and p; Dan was surprised to see that the end of the altitude M was in fact not on but above the line p, and reacted by saying: "I need to move k."



He looked for a command to move a line but obviously couldn't find one. This led him to the conclusion that he needs to construct a parallel to AC through M.

c) In contrast to Dan, Ruth started by analyzing the problem and concluded that after copying the segment AC she needs to construct a parallel to AC at the distance h. Thus she typed PA for parallel and learned from the program's requests that constructing a parallel at a given distance is not a basic construction. A search in the list of commands helped her find the sequence of three commands which lead to the solution.

Discussion

Reactions that were observed during the solution of construction problems with the microworld clearly show that the program is indeed a natural corrective system. The observations indicate that corrections occur with respect to the three levels mentioned before.

The analytical level: An incorrect problem analysis leads to a conflict between student expectations and the actual picture on the screen. This conflict forces the student back to the analysis stage and thus to a correction of his plan. This behavior was observed again and again and is exemplified in observation (b) above.

The application of the rules of geometric constructions: By not being allowed to use any step that doesn't belong to the microworld, the student is forced to use only basic rules of geometric constructions. For example, see Dan's unsuccessful attempt to move the perpendicular in observation (b), or Ruth's trials in observation (c).

The syntactic level: The students learn the formal syntactic structure of geometric construction because the program requests and accepts only commands which are formulated in formally correct manner. In observation (a), for example, all students tried constructions which were not built according to the syntactic rules; they quite easily corrected themselves following the program's prompts.

In conclusion, the students spent most of their time and energy on the most important part of the solution process, the problem analysis. Our observations show that students who neglected the analysis stage were forced to stop and plan their solution strategy, whereas those who made an initially wrong analysis had to revise their strategy during the solution process.

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STRUCTURES AND PROCESSES IN THE ABILITY TO COMMUNICATE SPATIAL INFORMATION BY MEANS OF CODED ORTHOGONAL VIEWS

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Ewa Puchalska, University of Montreal

The study described here touches upon three different subjects: the development of spatial representation (geometry), the ability to communicate information by means of codes (semiotics), and structures and processes in the development of such abilities (cognitive development). Accordingly, it is clearly interdisciplinary in nature.

Levels and processes in spatial representation have been studied since Piaget and Inhelder's (1967) and Piaget, Inhelder & Szeminska's (1960) pioneering work. Small- and large-scale spaces, useful in geography, for instance, have been found to involve both establishing relationships between objects and the construction of frames of reference (Hart & Moore, 1973; Wapner, Cirillo and Baker, 1971).

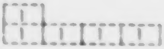
The use of codes forming a structured system, complying with conventions and rules of arrangement have been studied by Laborde (1982). It forms part of the (more general) ability called "graphicacy", which has been defined by Balchin (1972) as "the communication of spatial information that cannot be conveyed adequately by verbal or numerical means". Proficiency in graphicacy is considered by many as a fundamental goal of education for all, along with literacy, numeracy and oracy (Boardman, 1983).

Here the ability to code orthogonal projections of polycubical objects was studied. The first step was to build an ordinal scale of difficulty.

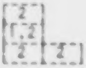
Method

After careful analysis, a theoretical ordinal scale was postulated, with eight levels of difficulty, and polycubical shapes were constructed purported to correspond to each of the levels. The shapes were the following, represented in orthogonal projection. Numbers inside the outline correspond to positions of cubes.

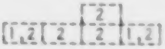
Orange 1
(preliminary)



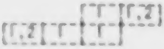
Orange 2
(preliminary)



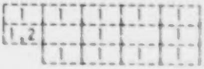
White



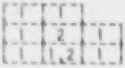
Light green



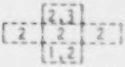
Blue



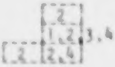
Yellow



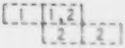
Black



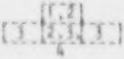
Red



Dark green



Purple



The task was administered in group form to students of eight classes from grades 4, 5, 6, 7, 9 and 11. Each subject received a set of the eight shapes, marked with a sticker to indicate the position in which it should be placed for observation. Then the subjects were instructed to do the following with each of the shapes successively: "(1) Keep it in front of you, at eye level, with the sticker placed at the bottom. (2) Draw the outline of the shape (on given squared paper) as it appears from that position. (3) Add information concerning other cubes, by means of signs placed in appropriate places. (4) Write out an explanation for each sign used. Your finished diagram must give an adequate representation of the shape, so that one of your friends could build it."

To ensure the understanding of the task, two preliminary orange items were introduced and discussed with the whole group, before the subjects started working individually. In order to guarantee the homogeneity of the technique, the same person gave the instructions in all classes, while two collaborators distributed the material and supervised the work.

The same three people did the marking of the huge number (more than 1500) of coded orthogonal views (COVs) obtained, first independently, then jointly. A COV was considered passed when it was possible to reconstruct the object from the diagram, with its codes and legends. Types of errors and coding strategies were noted on matrices representing frontal and sagittal layers of each object. The results obtained were categorized according to : (1) which items were passed and (2) which strategy or scheme was applied to code the objects. The quantitative results were submitted to an analysis of variance to determine the effect of items and age on performance. This was supplemented by scalogram analysis to investigate the hierarchic nature of items and a factor analysis was applied, by the Common factor method followed by varimax rotation, to determine the grouping of items according to common schemes or strategies put to use for solving them.

Results

Categorization - Categories of behavior were first established and subjects grouped according to the most elaborate each presented. An analysis of variance showed the influence of age was highly significant.

Scalogram - A scalogram was applied to items passed by each subject which showed that some items were too easy for subjects (white and light green shapes) and one too difficult (purple). However the plus percentage ratio PPR, independent of item difficulty, was found to be 0.71, above the 0.70 threshold usually accepted. Thus the scale of 8 items was found to be hierarchically ordered, though with some overlap.

Factor analysis - Factor analysis was then applied to the items scored as passed or failed, as it was surmised that common strategies in solving various items would lead to a correlation of success and failure and grouping of these items in a same factor (Noelting & Simpson, 1983). The Common factor method was used with varimax rotation (SAS, 1982) and a mineigen of .7. Factors were considered as independent, as the strategies applied to solve items, even if they are linked genetically, are different from one another.

Four factors were obtained, with loadings for each item on the factors given in Table 1. The corresponding levels of behavior which were ultimately established are given in the table.

Table 1. Factor analysis with varimax rotation (mineigen .7).

Shape	Factor 1	Factor 2	Factor 3	Factor 4	Level
White	0.09536	0.07472	0.0170	0.98852	1
Light green	0.81726	0.03916	0.10916	0.14418	2
Blue	0.82485	0.18363	0.10142	0.01349	
Yellow	0.78861	0.12551	0.19838	-0.00111	
Black	0.37518	0.03062	0.73771	0.03027	3
Red	0.03556	0.31701	0.80074	0.00061	
Dark green	0.14863	0.82865	0.12326	0.08794	4
Purple	0.11348	0.81879	0.16891	0.00590	

The results of the factor analysis were compared to the categories of behavior previously obtained and a close fit was found. This led to a final description of the levels of behavior found, both in terms of problem difficulty and coding stragegy. These levels are the following, with level 0 corresponding to total failure and level 5 to a new strategy in coding items which had already been passed at level 4.

Level 0 - Contiguity between cubes in the object is respected in the diagram, but without differentiation of the third dimension (depth). The object is rotated during exploration and cubes in the orthogonal axis are reproduced on paper as folded back.

Level 1 - Differentiation of the orthogonal axis and the plane of projection. The object is maintained in the same position during the coding process. Relations along the orthogonal axis are coded either in terms of order: "in front", or in terms of set: "double thickness". One direction only is found in the order relation, usually "in front". However, referent cubes in different orthogonal axes are not located on the same frontal layer: there is "lability" of reference. White shape only is passed.

Level 2 - Two directions are differentiated in the order relation: "in front" and "behind", with reference to the layer of cubes in the object presenting a flat surface. Light green, blue and yellow shapes are passed. The layer of reference, however, is not recognized inside an object with an uneven surface (red shape).

Level 3 - Constitution of a frontal layer of reference inside the object (black and red shapes are passed) with transitivity of positions along the orthogonal axes. However the layer of reference is limited inside the solid object and does not extend in space. When layers are "staggered", columns begin in different positions ("lability of referents").

Level 4 - An extended layer of reference is built, which includes exterior voids. There is thus stability of reference along the extended frontal layer, with possible readjustments in position through "mental leaps" when layers are staggered. Elements are coded both in terms of their nature (cube or void) and position (1st, 2nd, 3rd etc.). However these two aspects are still juxtaposed, leading to double coding of each element in terms of set and order.

Level 5 - A three-dimensional system of reference is built, resulting from its differentiation from the three-dimensional object. The two operations corresponding to set and order are differentiated and recombined at the level of each element, with a same symbol giving both the nature of the element and its position in the orthogonal axis (positional notation). However, two different coding systems are found, one centered on order, resting on the coordinate system built, with each position coded as filled or empty (e.g. 1, 0, 0, 1, 1); the other oriented on set and centered on the object, with the positions of each cube given (e.g. 1, 4, 5 for the same orthogonal axis).

These levels will be illustrated by examples.

Discussion

Halford (1978) had reinterpreted Piaget's stages in terms of three levels of complexity corresponding to the following mathematical structures: binary relations, binary operations and composition of binary operations.

A revised version of this model is presented, consisting in the five levels described, which lays a greater stress on process and integrates aspects from Piagetian theory with dialectics and systems analysis.

A STUDY OF THE APPLICATION OF A QUALITATIVE TAXONOMIC SYNTHESIS TO THE ANALYSIS OF GEOMETRIC REASONING IN A COMPUTER ENVIRONMENT

by

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INTRODUCTION

Recent national assessments of the mathematical ability of American high school students have highlighted a serious problem in the development of mathematical thinking: Students are failing in situations which require mathematical reasoning skills and understanding beyond the level of basic skills and memorization (Carpenter et al., 1983). The problem is particularly critical in high school geometry, where success depends on propositional thinking and deductive reasoning about geometric properties and relations.

The purpose of this research was to investigate the application of a taxonomic synthesis for assessing the potential of a computer learning environment, based on transformational geometry, for developing students' geometric reasoning skills.

A teaching experiment was designed to help ninth grade students progress through the levels of the SOLO (Structure of Observed Learning Outcomes) taxonomy in order to achieve a higher level of abstraction in their geometric thinking. The LOGO computer language was used as the vehicle for teaching and learning geometric properties and relations. The teaching methodology and curriculum ideas were based on a theory of relational learning cycles which emerged from a synthesis of the SOLO taxonomy and Skemp's model of mathematical understanding.

SOLO/Skemp Synthesis

The SOLO (Structure of Observed Learning Outcomes) Taxonomy (Biggs & Collis, 1982) was primarily designed as a tool for the evaluation of the quality of student responses to a task. The Taxonomy consists of five levels described as follows:

1. Prestructural: the learner does not engage the task or gives completely unassociated data.
2. Unistructural: the learner is able to use one piece of information only in responding to the task (formation of a single datum).
3. Multistructural: the learner is able to use several pieces of information but does not relate them together (acquisition of parallel data).
4. Relational: the learner integrates the separate pieces of information to produce a viable solution to the task.
5. Extended Abstract: the learner is able to derive a general principle from the integrated data which can be applied to new situations (acquisition of a higher-order concept).

Biggs and Collis's own integration of the SOLO levels into Piagetian developmental stage theory produced a model of learning cycles similar in many ways to Skemp's model of mathematical understanding and to particular aspects of Skemp's model of intelligence (1979).

Skemp (1976) proposed the existence of two types of mathematical understanding which could be generated by mathematics learning and teaching in schools: instrumental and relational understanding. Instrumental understanding is the product of rote learning of rules and theorems and their specific applications. Relational understanding is the product of a learner's personal involvement with mathematical objects, situations, problems and ideas.

[In order to avoid confusion at this point, between the different uses of the term "relational" in the two theories, the investigator has renamed the Relational level in the SOLO taxonomy the Relating level (the level at which data are interrelated).]

A synthesis of the SOLO and Skemp theories (Olive, 1983) suggests that learners acquire new understanding of subject matter by going through a learning cycle based on the SOLO Taxonomy in one of two ways: instrumentally or relationally. At each stage in a *relational learning cycle* (unistructural through extended abstract) the learner is personally involved with the available data. The data are, in fact, products of the learner's own investigations. In contrast, the data available in an *instrumental learning cycle* are given to the learner to memorize by some external source of information (usually the teacher, textbook or computer). The relating level of the SOLO cycle may also be force fed ("Here are the relationships - memorize them!") or may be omitted altogether ("Here is the general principle, or new theorem - memorize it!").

A Logo Learning Environment

Seymour Papert (1980), director of the MIT LOGO project, sets forth a thesis for the learning of school mathematics that is similar to that of Skemp. "What an individual can learn, and how he learns it, depends on what models he has available" (p. vii). Papert later makes a statement which emphasizes the importance of relational learning:

> Our educational culture gives mathematics learners scarce resources for making sense of what they are learning. As a result our children are forced to follow the very worst model for learning mathematics. This is the model of rote learning, where material is treated as meaningless; it is a *dissociated* model. (p. 47).

Papert and his colleagues created a new area of mathematics which they termed

"Turtle geometry," as part of the LOGO computer language. Papert claims that Turtle geometry is "a better, more meaningful first area of formal mathematics for children [than traditional geometry]" (p. 51) because it is associated with the child's view of the world and encourages reflective thinking in order to gain control of the world of the Turtle.

Turtle geometry has all the necessary ingredients for generating relational learning cycles: Being able to relate personally to a learning experience, being able to relate the experience to existing knowledge (and previous experiences), and reflection by the learner on what he/she knows. A review of previous LOGO research projects, however, indicates that LOGO has not been successful in generating relational learning for some students.

METHODOLOGY

Twenty students were randomly chosen from an intact, ninth grade class of 39 students in an urban high school. Each student worked with a micro-computer in a lab situation for 18 days (two hours a day, three days per week for six weeks). The investigator taught the group, introducing the students to the micro-computer and the LOGO language through a series of "guided discovery" learning episodes.

Each student's interactions with LOGO were saved on disk files. The data files of nine of the students were analyzed in depth in terms of the SOLO/Skemp synthesis. This analysis provided a picture of these students' developmental growth in the use of LOGO and helped to determine the appropriateness of the teaching methodology and curriculum ideas for generating relational learning cycles and helping students achieve a higher level of mathematical abstraction.

RESULTS

The results of the analysis indicate that for several students, the instructional sequence was too fast. There was not enough time for them to explore new programming ideas or to investigate the various geometric relationships before new ones were introduced. Consequently, their understanding of both the LOGO language and the geometric concepts was generally *instrumental*. However, for those students who were able to keep pace with the instruction, progression through SOLO learning cycles was evident. These students demonstrated a shift to a more abstract mode of functioning with the LOGO language and *relational* understanding of many of the geometric concepts that were introduced.

The results also demonstrate the enormous potential for process analysis provided by the data files. The ability to capture every move a student makes while working on a problem and recreate its effect visually on the computer brings us closer to observing directly the dynamic development of students' cognitive processes.

IMPLICATIONS FOR INSTRUCTION

The analysis of the data files also enabled the investigator to identify the gaps in the instructional sequence. These gaps highlighted the critical importance for introducing ideas at the appropriate SOLO level for individual students, for sequencing activities according to a SOLO cycle, and for encouraging reflection by the students on emerging relationships. The instructional sequence has been modified to better reflect these characteristics. An example of such a modification will be demonstrated at the conference.

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A Clinical Investigation of the Impact of a LOGO Learning Environment on Students' van Hiele Levels of Geometric Understanding

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Problem Statement

There is a national concern in the United States for the number of students who are failing to complete their studies of mathematics in high school successfully. Because less than one third of American high-school students take mathematics courses beyond tenth grade geometry (NASSP, 1983), it is clear that high school geometry is a critical turning point for many students in their mathematics studies. Failure in, or avoidance of, geometry results in the termination of the study of mathematics for the majority of students.

Since the curriculum development era of the 1960's, geometry has been introduced into the school mathematics curriculum in the elementary and middle school grades. However, geometry instruction in the early grades usually consists of identifying shapes and stating properties of geometric figures at a recall level without attending to more complex concepts that require understanding of relationships among geometric figures and their properties. The National Assessment of Educational Progress (NAEP) indicated that:

> Although most students at all age levels [9, 13, and 17 year olds were tested] could identify common geometric shapes, relatively few demonstrated a knowledge of basic properties of these shapes. ... Fewer than 20 percent of the 17-year-olds could calculate the area of either a triangle or a parallelogram. (Carpenter et al., 1983, p. 655)

Mathematics learning theorists agree that the understanding of relationships is the key to success in formal studies of geometry. These relationships include relationships among geometric figures, relationships among their properties, and even relationships among their relations.

The van Hiele Model

The van Hiele model of thought development in geometry has been used by mathematics researchers to explain why students have difficulty with more complex concepts in geometry because it partitions the learning of geometry into five distinct levels. The following is a description of the hierarchy:

> The first level is one of recognition of geometric figures as entities, without any awareness of parts of figures or relationships between components of the figure. In the second level, the components of a figure can be discerned and the properties of a figure described but not formally defined. Also a student at this level may recognize that two figures have properties in common but s/he does not conclude, for instance, that a rectangle is also a parallelogram.
>
> In the third level, relationships among the properties of a figure are established as are relationships among the figures themselves. The possibility of one property following from another is known, and a logical partial ordering of classes of figures occurs. It is only in the fourth level, however, that the logical structures of analysis and of proof are grasped and that deduction is understood as a means of constructing a geometric theory. The fifth level is characterized by standards of rigor and abstraction represented by modern geometries. A student at this level develops a geometric theory without reference to concrete applications. (Skypek, 1982, pp. 1-2)

Certain characteristics accompany the van Hiele levels and were described by Usiskin (1982):

a) The levels exist in a fixed sequence; one must pass through level n to attain level n+1;

b) Each level has its own language so that two people who are reasoning on different levels may not understand each other;

c) The levels have the property of adjacency; That which was an object of perception at the lower level becomes the object of thought at the next higher level.

Since one of the qualities of the van Hiele model is that students must pass through the levels in a fixed sequence, one must question whether students in high school geometry courses have achieved the first, second and third levels prior to their formal axiomatic studies.

The Curricular Gap at Relations

Evidence exists to support an assertion that relations are not being taught adequately in American elementary and middle schools. Wirszup (1976) was the first person to describe the van Hiele model in an American journal in which he called for geometry curriculum reform.

> As a result of unsuccessful experience and convincing evidence, the so-called axiomatic methods of initiation into geometry have been recognized by modern educators the world over as unpedagogical. A review of the teaching of geometry in the United States indicates at once that only a very small number of the elementary schools offer any organized studies in visual geometry, and where they are done, they begin with measurements and other concepts which correspond to Levels II and III of thought development in geometry. Since level I is passed over, the material that is taught even in these schools does not promote any deeper understanding and is soon completely forgotten. Then, in the 10th grade, 15 and 16 year old youngsters are confronted with geometry for almost the first time in their lives. The whole unknown and complex world of plane and space is given them in a passive axiomatic or pseudo-axiomatic treatment. The majority of our high school students are at the *first* level of development in geometry, while the course they take demands the *fourth* level of thought. It is no wonder that high school graduates have hardly any knowledge of geometry, and that this irreparable deficiency haunts them continually later on. (p. 96)

Geddes (1982) supported this finding in an analysis of elementary mathematics textbooks in which she found that most of the textbook material corresponds to the first van Hiele level. She notes that texts provide little support materials for moving on to level two, even in the teacher guides. "The lack of extensive [2nd] level experience in grades K - 8 indicates that many students enter high school geometry courses (which require [3rd] level thought) with a [1st] level background" (p. 22).

Bridging the Gap

This research study is attempting to bridge the gap between the middle school study of shapes and properties and the tenth grade treatment of formal deductive geometry by providing ninth grade students with a better understanding of geometric relationships. The LOGO computer language is being used as the vehicle for providing this understanding.

LOGO, through its graphics capabilities, embodies a physical robot, called the Turtle, which is a transitional object for the learner. The Turtle provides specific mental models for and experiences with those geometric relationships which cause problems in the present school curriculum, i.e. incidence, parallelism, perpendicularity, proportion, and angular measure. Exploring Turtle Geometry gives the student an opportunity to relate personally to angle measure, relationships between various geometric figures and their properties, and all geometric transformations. This provides the opportunity for meaningful reflection on mathematical ideas.

Design of the Study

For each of the past three semesters a LOGO course, based on Turtle geometry, has been offered in two inner-city schools. Ninth grade students who were tracked in an average-paced curriculum, who were enrolled in algebra, and who were scheduled to take geometry in the tenth grade were eligible to enroll in the LOGO course for one semester. The subject pool consisted primarily of minority students. Comparison groups of students, who were not enrolled in LOGO, were identified in each school each semester. The purpose of the study is to assess, within the framework of the van Hiele model, whether students' geometric concepts are enhanced by experience in a LOGO learning environment.

The Clinical Interview

How can these conceptions be assessed? Fuys and Geddes (1984) suggest that "conventional tests or assessments of level of thinking may not adequately characterize the student's ability to think at certain [van Hiele] levels, especially when there has been little or no opportunity to experience topics ... in school" (p. 11). Furthermore, "this more dynamic form of assessment during a learning experience, as Dina van Hiele-Geldof did in her teaching experiment, enabled [us] to examine changes in a student's thinking, within a level or to a higher level, and also difficulties which impeded progress" (p. 6).

The disinction must be made between static and dynamic assessments. In a static assessment, a student's misconception could be labeled **wrong**. In a dynamic assessment, where the interviewer has an opportunity to probe, a student's misconception might be labeled **alternative conception**. These alternative conceptions often lead to the most revealing understandings of students' mathematics.

This research project will involve the identification of cognitive activities (levels, in this case). For such an undertaking the clinical interview is structured to lead the subject's activity into particular areas of investigation. In the case of this research the same basic script is administered to each subject. Ginsburg (1981) cited the **contingency of questioning** as the one major element which sepsarated clinical interviews from standardized tests. The interviewer's questions are contingent upon the subject's previous responses to other questions; thus the instrument incorporates branching. Another important feature of the clinical interview is that reflection is required on the part of the student. One asks **how** the child arrived at a certain answer (process) and **why** he got a particular solution (product); thus probing is an important aspect of the interview.

Interview scripts and protocols for analyses were used for two topics: quadrilaterals and angles. The former was adapted from an instrument developed by Burger and his colleagues (1982) and the latter was developed by the project staff, piloted, validated, and revised. The instruments are being used to interview a sample of approximately 20 students before and after their LOGO experience and a comparison group of about 20 non-LOGO students is being interviewed likewise. The interviews are given in one-to-one sessions of 45 to 60 minutes and are audiotaped. Those audiotapes are currently being analyzed for evidence of treatment effect.

Results

Preliminary analyses indicate that these students are at the first or second van Hiele level on their initial interview. Angle item analyses point to discrepancies between student's understanding of static angle and their ability to apply that knowledge to tasks that involve turning angle. Several students who identified a simple drawing of a right angle as measuring 90° could not successfully turn a spinning arrow through 90° to aim it at a target. Many students were distracted in their ability to distinguish between a larger and a smaller angle by the irrelevant attribute of represented length of the rays of the angles. Activities are being designed and incorporated into the Turtle geometry course specifically to address these problems. Analyses of the second (post) interviews will be used to determine whether experience in the LOGO environment has any effect on students' understanding of geometric relationships. It is believed that the clinical interview is specific and sensitive enough to detect both movement within van Hiele levels and between levels.

Implications

Results of final analyses should have important implications for understanding student thought processes in geometry and could, in turn, impact on curricula development in the U.S.A. Samples from the interviews and all analyses completed to date will be presented at the conference. It is hoped that the results will contribute to PME's goal of furthering a deeper and more correct understanding of the psychological aspects of teaching and learning mathematics.

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An Audio-Visual Test for Promotion of the Thinking Ability and Its Application to Math Classes

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1. Introduction

In an ordinary math class, the response each student shows to the audio-visual information given by the teacher is peculiar to the student. However, it is very difficult for the teacher to cope with all different responses shown by each student. Then the teaching-learning process is too difficult for one group of students and is very easy for another group of students. Therefore, it is rare that the math class satisfies the following conditions.

(1.1) Every student always pay keen attention all through the class.
(1.2) The most suitable difficulty is given to each student.
(1.3) Every student (even slow learner) is able to develop his power of thinking at his own pace in the teaching-learning process.

We have developed two new Audio-Visual Tests (AV-Tests), Cloze-Type AV-Test (T_{AV}^{C}), Jigsaw Puzzle-Type AV-Test (T_{AV}^{J}), and by introducing T_{AV}^{C} and T_{AV}^{J} into the teaching-learning process, we were able to establish a situation which satisfies (1.1-1.3). Main purpose of this paper is to show the situation.

(**Remark**) Cloze Test is the test used to evaluate language proficiency.

2. Intellectual Hunger Situation

It must be remembered that many of the discoveries and inventions were made by people who were anxious to be intellectually(or materialy) satisfied. If we emphasize this fact, the student will bring their intellectual ability into full play under the condition that they are anxious to be intellectually satisfied in the teaching-learning process. Now, let us find the situation where the student's intellectual activity is in full activity. It is sufficient to consider the following cases:

Case 1) It seems that the student is able to understand the teaching-learning process,even though enough information is not yet given to him.

Case 2) It seems that the student is able to understand the teaching-learn-process, since enough information is given to him.

Case 3) It seems that the student is not able to understand the teaching-learning process on account of the insufficiency of information.

Case 4) It seems that the student is not able to understand the teaching-learning process, even though enough information is given to him.

In ordinary math class, when it seems that the student is not able to understand the teaching-learning process, his learning attitude becomes to worse. Namely, his intellectual activity is not in full activity in cases 3) and 4). Obviously, it is more difficult for the students to understand the teaching-learning process in case 1) than in case 2), because, in case 1), the student is always anxious to obtain new information which is usefull for him to understand the teaching-learning process. That is to say, in case 1), the student is moderately hungry for the information in the teaching-learning process. So we call the teaching-learning process in case 1) " Intellectual Hunger Situation ".

We have developed a Intellectual Hunger Situation using T_{AV}^{C} and T_{Av}^{J} in order to design the math class which satisfies the conditions(1.1-1.3) First, we will briefly state about T_{AV}^{C} and T_{AV}^{J}.

2. Cloze-Type AV-Test T_{AV}^{C} and Jigsaw Puzzle-Type AV-Test T_{AV}^{J}

T_{AV}^{C} : T_{AV}^{C} is made and put into practice in the following way.

(2.1) We write down the protocol on the bases of the teaching-learning process.

(2.2) We give the students audio-visual information in accordance with the protocol in 2.1).

(2.3) Combining the audio-information with the visual infomation given by the teacher the students take notes on the teaching-learning process.

(2.4) Referring to the notes on the teaching-learning process, the students fill in the blanks of the Cloze Test on the protocol.

T_{AV}^{J} : T_{AV}^{J} is made and put into practice in the following way.

(2.5) We illustrate the teaching-learning process with the diagram along the protocol in (2.1).

(2.6) We make a Jigsaw Puzzle from the picture drawn in (2.5).

(2.7) We give the students audio-visual information in accordance with the protocol in (2.1).

(2.8) Combining the audio-visual information given by the teacher, the students take notes on the teaching-learning process.

(2.9) Referring to the notes on the teaching-learning process, the students fill in the blanks of the Jigsaw Puzzle in (2.6).

3. An AV-Test for Promotion of the Thinking Ability

In an ordinary math class, it is necessary for the students to combine the visual information obtained from the text, OHP and blackboard with the acoustic image corresponding to the teacher's explanation in order to understand the teaching-learning process. Further, in order to combine the audio information with the visual information, translation of the presentation-mode of the concept is requisite for the students.

(Remark) The presentation-mode of the concept: Aural-mode, Diagram-mode. Ordinary mathematical concepts are able to be represented by use of one of these two modes.

We have developed an AV-Test (T_{AV}) aiming at the correct combination of visual information with acoustic image in mind. (See (4) of the reference at the end of the paper.) In T_{AV}, each student's memory-span of audio(or visual) information given by the teacher was peculiar to the student. Moreover, in traslation of the presentation-mode, each student's time required was also pequliar to the student. Therefore, in many cases, the explanation of the concepts(or problem-solving process) given by audio-visual information set in T_{AV} was too difficult for one group of students and very easy for another group of students. This means that it is very difficult for the teacher to convey a large-scaled-concept one-sidedly only by audio-visual information. However, in T_{AV}, we found that all students always pay keen attention. From the above-mentioned, we might conclude that the AV-Test (for a example T_{AV}) aiming at the correct combination of visual information with acoustic image in mind is very suitable for the teaching-learning process where the small-scaled-concepts are treated.

In consideration of the above, by using T_{AV}^{C} and T_{AV}^{J} , we have designed a teaching-learning process which satisfies the following:

(3.1) The teacher gives the students audio-visual information only about the small-scaled-concepts.

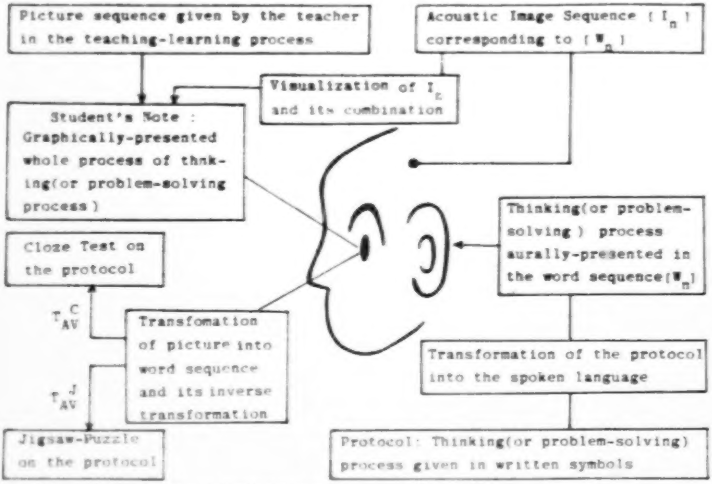
(3.2) Cmbining the audio information with the visual information, the students take notes on the concepts given in (3.1).

(3.3) Referring to the notes in (3.2) and hints given by the teacher, the students are able to construct(or understand) a large-scaled concept from the concepts in (3.1) at their own paces.

In order to satisfy these three conditions, we combine T_{AV}^{C} with T_{AV}^{J} in the teaching-learning process.

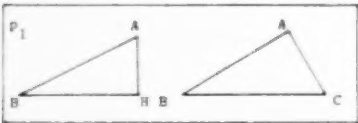
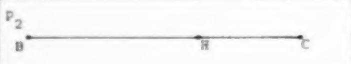
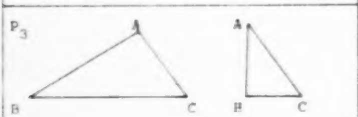
We show an example of the teaching-learning process using T_{AV}^{C} and T_{AV}^{J} . We call the teaching-learning process " ($T_{AV}^{C} + T_{AV}^{J}$)-Type AV-Test". The flow diagram for ($T_{AV}^{C}+T_{AV}^{J}$)-Type AV-Test is shown in (Fig-3.1). In ($T_{AV}^{C} + T_{AV}^{J}$)-Type AV-Test,we give the students the small-scaled-concepts at random by television. Therefore, the students are not able to consider what those concepts mean as a whole. Of course, the students may ask the teacher about the audio-visual information given by the television.

(Fig-3.1) Flow Diagram for $(T_{AV}^{C}\cdot T_{AV}^{J})$-Type AV-Test


Picture sequence given by the teacher in the teaching-learning process
Acoustic Image Sequence $\{I_n\}$ corresponding to $\{W_n\}$
Visualization of I_n and its combination
Student's Note : Graphically-presented whole process of thnking(or problem-solving process)
Thinking(or problem-solving) process aurally-presented in the word sequence$\{W_n\}$
Cloze Test on the protocol
T_{AV}^{C}
Transfomation of picture into word sequence and its inverse transformation
T_{AV}^{J}
Transformation of the protocol into the spoken language
Jigsaw-Puzzle on the protocol
Protocol: Thinking(or problem-solving) process given in written symbols

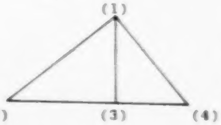
(Example : Proof of Pythagorean Theorem (Part•))

(Fig-3.2)

	W_1: The angle ABH equals the angle ABC. Therefore, $\triangle ABH$ is similar to $\triangle CBA$. Then we have $AB:BH=BC:AB$. Therefore, AB^2 equals BC times BH.
	W_2: The point H is on the segment BC. Then we have the following: BH plus HC equals BC.
	W_3: The angle ACB equals the angle ACH. Therefore, $\triangle ABC$ is similar to $\triangle HAC$. Then we have the folloing: $AC : HC = BC : AC$. Therefore, AC^2 equals BC times HC.

.... T_{AV}^{C}........... (Fig - 3.3) ....T_{AV}^{J} ......

On the right triangle(1), the angle BAC is the right angle and (2) is the foot of (3) on (4). On the right triangles (5) and ΔABC, the angle (6) equals the angle ABC. Therefore,(7) is similar to ΔABC. Then we have AB :(8) =(9) : AB. Therefore, (10)^2= (11)· BH. Similarly,since the right triangle ABC is similar to the triangle (12), we have AC : (13) = (14): AC. Therefore,(15)^{2}equals (16) times HC. Thus, we have the following: (17)^2+ AC^{2}equals (18)times BH plus (19)times HC. Since BH+HC equals (20), (21)times BH plus (22) times HC equals (23) times (BH+HC) equals (24)^2. Hence we have (25)^2 plus AC^2 equals (26)^2................



On ΔABH and (5),

∠AHB = (6) = (7)

∠ABH = (8).

Therefore, (9) = (10).

Hence AB/(10)=BC/(11),that is $(12)^2 = (13)\cdot BH$ ......(a)

Similarly, on (14) and (15)

∠AHC = (16) = (17)

∠ACH = (18)

Therefore, (19) = ΔABC.

Hence AC/(20)=BC/(21),that is $(22)^2 = (23)\cdot HC$. Thus we have $(24)^2+AC^2=(25)^2$

(Answer of T_{AV}^{C}) 1. ABC 2.H 3.A 4.BC 5.ABH 6.ABH 7.ΔABH 8.BH 9.BC 10.AB 11.BC 12.AHC 13.BC 14.BC 15.AC 16.BC 17.AB 18.BC 19.BC 20.BC 21.BC 22.BC 23.BC 24.BC 25.AB 26.BC

(Answer of T_{AV}^{J}) 1.A 2.B 3.H 4.C 5.ΔABC 6. ∠BAC 7. ∠R 8. ∠ABC 9.ΔABH 10.BH 11. AB 12.AB 13.BC 14.ΔAHC 15.ΔABC 16. ∠BAC 17. R 18. ∠ACB 19.ΔAHC 20.HC 21.AC 22.AC 23.BC 24.AB 25.BC ....

(Step 1) We give the students audio-visual information as shown in (Fig-3.2). Cmbining audio-information with visual-information, the students take notes on the teaching-learning process.

*** In Fig-3.2, W_n and $P_n(n=1,2,3)$ denote the teacher's voice and the picture on the television screen respectively.

(Step 2) Referring to the notes, the students fill in the blanks of T_{AV}^{C} and T_{AV}^{J} in (Fig-3.3).

(Remark) The students may check their answer by removing the covering on the blank , if they want.

4. Some Consideration on the Teaching-Learning Process Using T_{AV}^{C} and T_{AV}^{J}

The teaching-learning process using T_{AV}^{C} and T_{AV}^{J} has the following characteristics.

(4.1) From (Fig-3.1) , it is easy to see that the teaching-learning process using T_{AV}^{C} and T_{AV}^{J} is a combination of the test with learning.

Therefore, the students always develop their thinking unconsciously in suitable Test-Situation.

(4.2) Obviously, the student's note in (Fig-3.1) is a kind of set of right answers to T_{AV}^{C} and T_{AV}^{J}, so each student is confident that he can fill in the blanks of T_{AV}^{C} and T_{AV}^{J} with the right answers. But the students must seek the information corresponding to the blank. Namely, the students are hungry for the information. Therefore, we may regard $(T_{AV}^{C}+T_{AV}^{J})$-Type AV-Test as a Intellectual Hunger Situation.

(4.3) We tried to give student k his score(X_k) of $(T_{AV}^{C}+T_{AV}^{J})$-Test in the following way. X_k=(number of the right answers)$\cdot(1-N/N_k)$, where N is the number of coverings, N_k is the number of coverings removed by k. Then the students showed a tendency to arrange their thinking very carefully.

(4.4) Since the students may check their answer by removing the covering, if they want, they are always able to develop their thinking ability at their own paces in $(T_{AV}^{C}+T_{AV}^{J})$-Type AV-Test.

Thus we might conclude that $(T_{AV}^{C}+T_{AV}^{J})$-Type AV-Test satisfies the conditions (1.1 - 1.3).

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3. DEVELOPING AND/OR USING MODELS OF MATHEMATICAL LEARNING

Developing a Model to describe the Mathematical Progress of Secondary School Students (11-16 years): Findings of the GRADED ASSESSMENT IN MATHEMATICS Project.

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Introduction

The GRADED ASSESSMENT IN MATHEMATICS (GAIM) Project is developing a method of recording mathematical achievement for all children throughout the 11-16 age range. The student profile covers many different aspects of mathematics; in particular 'process' and 'content' are seen as inextricably linked.

The purpose of the record is

- to enable students to participate more effectively in determining the direction of their own learning;
- to motivate students by making them more aware of their own progress;
- to provide full diagnostic information for teachers which can serve as a basis for curriculum planning;
- to provide information, summarised to whatever degree is required for parents, employers, training schemes, further and higher education, head teachers, and so on.

One major problem was to determine the model of progressive achievement to be used for the record. An overall structure for this model was agreed during 1983-4; since then detailed sections of the structure have been proposed and tested empirically. This paper will concentrate on this issue, although many other important questions concern the project, such as whether the structure can be adequately communicated to teachers, children, parents and users, whether the proposed assessment and recording processes are both feasible and valid, whether the four aims expressed above are being fulfilled, and so on.

A model for assessment

The cognitive theory adopted is a constructivist one in which children progressively assemble and modify their mathematical schemes following the ideas of Piaget (1980), Ausubel et al. 1978), von Glasersfeld (1981). Hence each child is likely to have a unique sequence of mathematical development. Nevertheless it is suggested that within particular concept strands children tend to follow a similar developmental sequence, although progress in different concept strands is likely to be to some extent independent. It is therefore postulated that there are enough similarities between children's developmental patterns to assume an underlying model of a partially ordered common hierarchy. Evidence for such a hierarchy is provided by previous research based at Chelsea College (now part of King's College London) such as that of Denvir (Denvir and Brown, 1986a, b) at primary level and of the Concepts in Secondary Mathematics and Science (CSMS) project at secondary level (Hart, 1981).

In order to simplify a complex and largely hypothetical underlying learning hierarchy for the purposes of assessment, recording and reporting, it was decided to organise the assessment framework as a profile with two dimensions - a number of progressive levels and six topic areas (Logic, Number, Measurement, Statistics, Space, Algebra/Functions). Achievement at a particular level within a particular topic will be described by a number of detailed 'topic criteria'. The topic criteria will generally be demonstrated by performance in open-ended work, either practical problem-solving or an investigation.

The following problems remained:

(a) to determine how many levels should be used;

(b) to determine what topic criteria would be appropriate for each level.

How many levels?

The CSMS project referred to above identified progressive levels of attainment (between 4 and 7) in each of 11 mathematical concept-areas (Hart,1981). The proportion of children attaining each level in each CSMS concept-area was plotted across the 11-16 age range as in Fig. 1. (Fig 1 also shows the classification of the various levels into four broad stages.

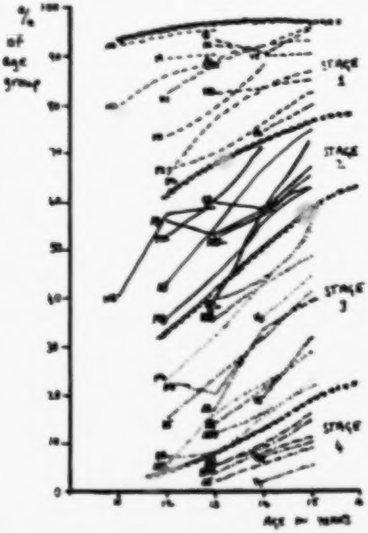

Fig. 1

The percentage of children at each level in each CSMS topic across the 11-16 age range (data from Hart, 1981)

Key A: Algebra D: Place Value & decimals F: Fractions G: Graphs M: Measurement N: Number operations R: Ratio RR: Rotation and reflection V: Vectors.

A2 shows the percentage of children attaining level 2 in algebra, etc.

In spite of some variations between concept areas perhaps due to the different points at which they are generally introduced into the curriculum, it is interesting to note both the overall similarity in trends and the generally low gradients. Indeed if the curves were to be extrapolated (a procedure of doubtful validity) it would suggest a delay of well over 10 years between the ages at which high-attaining and low-attaining children achieve the same level, in contrast to the 7-year difference proposed in the Cockcroft Report (Committee of Inquiry, 1982, p342).

A possible organisation for the GAIM assessment framework was offered by the four overall CSMS stages, but the above diagram suggests that it would then take many years for a child to progress from one stage to another, which would be a de-motivating factor. Hence it was decided to:

(a) adopt enough levels to on average allow students to progress through one level per year

(b) define level 1 as that level achieved by all children in mainstream education by the end of year 3 (age 14)

(c) further adjust the later seven levels so as to match the known percentages of children gaining the seven grades in the present national public examinations at 16+ (GCE O-Level or CSE).

This suggested a total of 15 levels for the GAIM assessment scheme with a theoretical distribution as in Fig. 2, based on the CSMS graphs in Fig. 1.

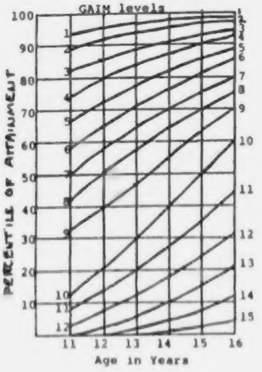

Fig. 2

Predicted percentage of children achieving each GAIM level across the 11-16 age range

What topic criteria at each level?

An outline of what was involved in each topic area was proposed, including general processes, concepts, representations, skills and applications.

Survey results from the Assessment of Performance Unit (1985) and CSMS (Hart, Brown et al, 1985) enabled some topic criteria to be allocated to specific levels by using Fig. 2. (This was not straightforward since the percentage of children who have achieved a particular criteria is clearly not identical to the percentage who correctly answer a specific item.) For example the measurement criterion:

> Knows that when a larger unit is used the numerical value will be smaller than if a smaller unit is used.

could be tentatively placed at level 3 using the results of item 2 on the CSMS Measurement Test.

However very many of the suggested topic criteria still remained to be placed. In such cases a conjecture is made using a mixture of teacher judgement and reference to curricular schemes empirically organised into levels, such as the Secondary Mathematics Independent Learning (SMILE) scheme and the Kent Mathematics Project (KMP). The conjectures are then tested out empirically, first in development trials and later in 25 pilot schools. This is a fairly crude process since neither the assessment procedures nor the identification of the attainment-range of classes within the overall population would meet the requirements of rigorous research. Nevertheless this 'action research' within the limits of what is possible in a development project with a tight schedule, will produce a comprehensive picture of children's broad mathematical development over the 11-16 age-range.

At present levels 1-3 are developed and piloted, and levels 4-8 are developed but not yet piloted. The project is due to finish in 1989.

Some examples of tentative placements of topic criteria are given below:

Level 1, Logic: Can use and understand a single system of ordering to locate or place an item.

Level 7, Statistics: Can choose to find a representative value (mean, mode, median or some other) to solve a problem, for example in order to compare two populations.

Level 2, Space: Can draw and interpret 2-D representations of familiar objects or scenes. These need not strictly be plans, nor to scale.

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An Investigation into the Sensory-motor and Conceptual Origins of the Basic Addition Facts

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This study was premised on the belief that mathematical knowledge is constructed by reorganizing sensory-motor and conceptual activity. It was hypothesized that children's learning of the basic addition facts is related to qualitative conceptual change both in cases where facts are learned as a family and when individual facts are learned separately. Steffe et al.'s (1983) counting types model was used to infer the subjects' conceptual levels. This model identifies five qualitatively distinct levels in counting but, for the purposes of this paper, it suffices to consider the distinction between the first four and the fifth level. The more sophisticated abstract stage is strongly indicated by the ability to routinely count to solve missing addend tasks. A child might, for example, solve the sentence 8 +__ = 12 by counting "8 - 9, 10, 11, 12 - 4." The crucial feature of such solutions is that the child formulates the intention of finding out how many counting acts he or she will perform when counting beyond "eight" to "twelve." This indicates that abstract counters can reflect on a re-presentation of potential counting activity. Children at the lower enactive stage find missing addend tasks difficult because they are yet to develop this ability.

The distinction between counting from "one" and counting-on corresponds only very roughly to that between the enactive and abstract stages in counting. Some children within the most advanced level within the enactive stage (i.e. verbal counters) count-on beyond a re-presentation of the activity of counting from "one." These children are infered to be "inside" the re-presented counting activity -- they can create a re-presentation but cannot reflect on the result of the re-presentational activity.

The specific hypotheses addressed were:

1. The learning of the plus-one fact family (e.g. 2+1, 5+1) is a consequence of the ability to create a conceptual referent for the first addend without actually having to count from "one." In other words, the child can count-on.

2. The process of learning individual facts separately involves anticipating the conceptual results of activities that could be carried to solve the facts. These anticipations depend on the ability to reflect on potential activity -- the child is an abstract counter.

(A hypothesis concerning the doubles facts is omitted due to space limitations).

Method

Video-taped clinical interviews were conducted with 15 beginning first graders who were yet to be trained to memorize the addition facts in school. Addition and missing addend counting tasks involving visible and screened collections (cf. Steffe et. al., 1983) were administered to infer the child's conceptual level and whether he or she could count-on. Basic fact tasks were presented in the following order: 2+1, 4+1, 3+1, 5+1, 2+2, 5+5, 4+4, 3+3, 4+2, 3+2, 5+2, 5+4, 4+3. These facts were chosen because they can be solved by using the relatively primitive method of establishing a finger pattern for each addend on separate hands. Finally, number word tasks were presented to investigate how the child generated the first successor and the first two successors of given number words. The child was asked, "Which number comes right after n?" and "Which two numbers come right after n?" for at least two trials with n less than ten.

Findings

Counting tasks. Eight of the children were classified as enactive counters and seven as abstract counters. One of the eight enactive counters and five of the seven abstract counters counted-on to find sums.

Number word tasks. All 15 children generated both the first successor and the first two successors of given number words that preceeded "ten" without reciting starting at "one."

Plus-one facts. The children's performance on the plus-one facts is summarized below. The methods subsumed under "Counted from one" ranged from counting the fingers of two finger patterns to subvocally uttering number words. As can be seen, there is an extremely strong relationship between counting-on and knowing the plus-one facts that were administered. All nine children who counted from one were able to generate immediate successors of number words that preceded "ten" with ease. This strongly indicates that knowledge of the plus-one facts is not derived from the child's ability to operate on the forward number word sequence. Instead, it is derived from counting, a process in which each number word signifies a unit of some kind. For our subjects, to know the n+1 facts is to be able to construct a conceptual entity signified by "n" without have to count. Consequently, they answered by uttering the immediate successor of "n", and act that carried the significance of performing one counting-on act.

	Counted on at least 3 of 4 facts	Knew all four facts
Counted from one	9	0
Counted-on	0	6

Remaining facts. The results are summarized in the table below. In general, knowledge of these facts is related to the ability to abstractly count-on.

	Number of Facts Known				
	0-1	2-3	4-5	6-7	8-9
Enactive- Counted from one	5	1	1	0	0
Enactive - Counted-on	0	0	1	0	0
Abstract - Counted from one	1	1	0	0	0
Abstract - Counted-on	0	0	2	0	3

However, one enactive counter who counted from one knew five facts and the enactive counter who counted-on knew four facts. Significantly, both children knew all four doubles facts. One aspect of their problem solving activity differentiates them from the other six enactive counters. Both children's performance on spatial pattern and spatial visualization tasks not reported in this paper indicated that they had exceptional spatial abilities. They might therefore have solved the doubles facts by re-presenting and figurally joining two patterns.

It will be recalled that abstract counters are attributed the ability to reflect on potential counting activity. Consequently, the five abstract counters who counted-on might have learned the facts by estimating or gauging where they would stop counting if they were to, say, perform three counting acts beyond "four." In other words, they could anticipate the conceptual result of counting and so could construct the conceptual entity corresponding to the sum without actually having to count.

It only remains to explain why the two abstract counters who counted from "one" knew, at most, two of the facts. The most plausible explanation is that these children could not reflect on potential counting activity unless they were actually counting. The had to actually count from "one" to construct a conceptual entity corresponding to the first addend but could reflect on a continuation of that activity. In contrast, the five abstract children who counted-on did not need a "running start" but could take the first addend as a given (i.e. construct it conceptually) and thus anticipate the conceptual result of counting.

Discussion

The findings indicate that both the learning of facts by families (e.g. the plus-one facts) and the separate learning of individual facts are related to conceptual development -- to the meanings children give to addition tasks. With regard to the learning of families, the findings are consistent with the view that constructing mathematical relationships involves abstracting from and reorganizing sensory-motor and conceptual activity. The emphasis on activity is compatible with an analysis of children's construction of thinking strategies to find sums and differences (Cobb, 1983). This view can be contrasted with the contention that constructing relationships involves internalizing increasingly complex rules and principles that the adult observer can "see" in the child's environment (cf. Baroody, 1985).

The finding that the process of separately learning individual facts is also related to the child's meaning-making activity challenges associationist models proposed by Ashcraft (1983) and Siegler and Shrager (1984). In particulary, the hypothesis that learning facts separately involves incrementallly increasing the association between the stimulus and the answer the child computes appears to be a gross over-simplification.

The most important pedagogical implication is that enactive counters should not be drilled on the basic facts even if learning the facts is given priority.

Although drill-based instruction might be successful with some abstract counters, considerable evidence indicates that it will be more profitable to encourage the construction of thinking strategies. And these strategies have as a conceptual prerequisite the attainment of the abstract stage in counting (Cobb, 1983).

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THE COGNITIVE DISTANCE BETWEEN MATERIAL ACTIONS AND MATHEMATICAL OPERATIONS

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We are concerned with the relationship between concrete manipulative actions and (certain) mathematical operations. It is commonly accepted and asserted that this relationship is a close one and that it could and should be exploited for didactical purposes. For the cognitive development of an individual the actions and manipulations are assumed to be a primary stage out of which and based on which mathematical operations will grow as a kind of secondary stage or level. Thereby usually and implicitly a more or less smooth transformation or transition from the outside world (material actions) to the inside world (mathematical operations as mental actions) is postulated. It appears as if this transition were considered as an almost automatic process and that it sufficed therefore to let the learner carry out the material actions. According to such a position the mathematical operation in a way is contained in the material actions and can be isolated out of it by a process of abstraction which leaves aside the accidental, irrelevant and specific features. This cognitive process has been termed in different ways; abstraction from the action, reflective abstraction, interiorization, schematization and others.

Contrary to that in this paper a constructivist position on the relationship (where it exists at all) material action - mathematical operation will be proposed. This position is based on the general principles of constructivism, see for instance [5]. But there are other sources of evidence from which I draw my conclusions. The first is the cognitive and psychological status of what usually is called a mathematical object (like numbers, geometric forms, functions and so on). These are mental entities or constructs characterized by certain properties, by relations among each other and by relations to the material world; compare for this the "conceptual entities" in [6]. Usually these entities are designated by names or mathematical symbols. What is important here is that mathematical objects in this sense psychologically have a great degree of independence and in fact have an object-like character for our consciousness: we think about them, we talk about them, we feel their cognitive existence. Of course, these mental entities are the results of intensive learning processes and their cognitive reality is increased by mental reflections, i.e. it is the mental activity which constitutes mathematical objects in the cognition of the individual.

This description is compatible with the constructivist viewpoint and with every day experience as well. To use a term from cognitive psychology: mathematical objects correspond to (well developed) cognitive schemata or frames. This theoretical description gives some plausibility to an important feature of mathematical objects: the individual associates them with material objects as a kind of mathematical property and the respective schema/frame regulates this association. In other words: one knows which mathematical objects to associate with which material objects. Mathematical objects then turn out to represent systems of relations which are constituted by certain manipulations on or with the material objects (numbers for instance by counting or measuring), compare [2] and [3]. For the frames to comprise all this adequate learning activities are necessary which include the relevant manipulations of experiential objects.

To this psychological status of mathematical objects corresponds their epistemological status as objects in mathematical theories there termed as mathematical concepts. The use and treatment of mathematical concepts as members of theories underlines the independence and autonomy of mathematical objects.

This autonomy is further demonstrated by empirical findings on the mathematical behavior of students, see [7]. There a clear separation of mathematical objects from experiential objects is evidenced which extends to a similar separation of mathematical operations from material actions. This will partly be due to deficiencies in the learning process but might be caused by an inherent feature which is the central issue here and can be expressed as the following:

Thesis: Mathematical operations (whose operands are mathematical objects) are not simply contained as aspects or features in related concrete actions and cannot be isolated from them by simple abstraction. Mathematical operations have to be constructed mentally and to be integrated with the actions; they are not obtained by disregarding certain fearures but by adding additional features (the mathematical ones). I call this cognitive-constructive process constructive abstraction which results in a cognitive distance between actions and operations. On the other hand this construction is initiated, stimulated and controlled by certain aspects of the respective actions which permits then the individual associating operations and actions similar to associating mathematical and experiential objects. Mathematical objects and mathematical operations are mentally constructed in the context of concrete actions but with a definite independence of and distance to them. Therefore the actions do not completely determine the operations; different operations can be associated with the same type of action (and vice versa); of course there are operations which are not related to material actions at all.

In a way, mathematical objects and the operations with them constitute for each appropriately trained individual an independent mental realm which is connected and related to material counterparts by a rich manifold of associations constructed by the individual himself. Official mathematics is then the language for communicating on these realms and for formulating properties of its objects and operations.

If one takes this position serious then it will be no longer feasible to describe the relations between actions and operations as that of the special to the general or of the concrete to the abstract or of the external to the internal in the traditional meaning of these phrases. All these terms have in mind an original union of actions and operations which is broken up in the course of the (individual and social) development. For us there is no such a priori union but a complementary relationship which is of constructive nature and which is best termed as that of the mathematical to the material. It appears not to be feasible to term addition of rational numbers to be more general than sticking together rods since numbers and rods are incomparable with respect to the dimension "specific-general". Rods are not a subset of the set of (rational) numbers nor can they be viewed as such (like the natural numbers for instance). The example which follows will explain in more detail the general thesis and will give some insight in the nature of mathematical operations and how they are associated with material actions by constructive abstraction.

Example: Multiplication. This operation usually is introduced via actions like this: A child carries bottles from the cellar, 4 bottles at a time, and 5 times at all. How could one abstract multiplication 5x4 from this action by just leaving aside so-called accidentals? If you start from the action as you can observe it as an experiential process you never will arrive at 5x4 by abstracting from irrelevant features. Instead, one can explain the "apparent" connection of multiplying 5x4 with this action by the following constructive abstraction which mediates between the two qualitatively different levels of action and (mathematical) operation. First, attention has to be focussed on certain stages in the continuous flow of the action, compare [3]. We could say punctuation of the process has to be guided appropriately, compare [1]. Further with these stages and their elements numbers (more general: mathematical objects) have to be associated. For that to be possible, the appropriate cognitive schemata have to be developed by the individual. Specifically, here those stages are when the child arrives back from the cellar and the numbers of bottles are the relevant mathematical objects. Further, the number of visits to the cellar has to be counted. I should emphasize that this associating of numbers according to my viewpoint is a mental-constructive process which is guided or motivated by the action but which is not inherent in it. Similarly, the action can lead the cognitive attention to focus on the change of the numbers (i.e. exchange of mathematical objects) in the course of the action. This gives rise to the impression that the action changes the numbers (exchanges the mathematical objects) which makes sense only after having carried out mentally all the necessary associations of numbers to stages of the action. The next step of the mental construction is the mental establishment of relationships between the associated numbers (mathematical objects in a general context). Of course, there are many possibilities and just one of them is that between 5,4 and the total number 20 of bottles. One can say, that the action (and certain goals of it) via focussing attention induces the cognitive construction of the relationship or correspondence of 5 and 4 to 20 which is one between mathematical objects: $(5,4) \rightarrow 20$. It is important to note here that at this stage there is not yet any "operation" (in the form of an algorithm) on mathematical objects but just a (static) correspondence induced by the action and added to it in an integrative manner. The general form of this correspondence can be described as follow: If one carries m times k bottles then the total number of bottles carried corresponds to m and k or to (m,k). This correspondence can be presented by a table for instance as it was done before the invention of algorithms which permit the calculation of the total number out of m and k. For this to be possible appropriate symbols for the numbers have to be available (like place value systems). The algorithm (for multiplication here) can then be viewed as a (material) action to be carried out on the number symbols such that the same correspondence on the numbers (mathematical objects) is induced as it has been induced originally by the action. If one takes multiplication as a paradigmatic example this can be described in general by the following commutative diagram:

$$\begin{tikzcd}
\boxed{\text{material objects}} \arrow[rr, "\text{action}", "\text{transforms}"'] \arrow[d, leftrightarrow, "\text{constructive association}"'] & & \boxed{\text{material objects}} \arrow[d, leftrightarrow, "\text{constructive association}"] \\
\boxed{\text{mathematical objects}} \arrow[rr, dotted, "\text{induced}", "\text{relationship or correspondence}"'] \arrow[dd, leftrightarrow, "\text{mental association}"'] & {} & \boxed{\text{mathematical objects}} \arrow[dd, leftrightarrow, "\text{mental association}"] \\
& \text{"mathematical operation"} \arrow[u] \arrow[d] & \\
\boxed{\text{mathematical symbols}} \arrow[rr, "\text{algorithmic}", "\text{action}"'] & {} & \boxed{\text{mathematical symbols}}
\end{tikzcd}$$

For the developed thinking these qualitatively different levels or layers coincide since they will be integrated by the same frame/cognitive schema "multiplication". But the development of this frame needs intensive learning activities of the individual. Especially our analysis makes clear that the actions will not automatically and by themselves lead to the cogitive construction of the mathematical operation (or mathematical objects). Guidance of the learner is needed to attatin appropriate focus of attention on the relevant mathematical objects and their changes/exchanges. We mention that for this cognitive process the use of prototypes (didactical materials) is important which by their very structure and appearence enhance focus of attention on the respective mathematical objects (and their association with the prototypes).

Using nouns as names for mental entities is an important means for creating the subjective and cognitive creation of the object-character for such entities (like mathematical objects). I put forward the thesis that the use of verbs plays a similar role with respect to mathematical operations which as analyzed above at their genetic origin have rather the character of static correspondences. Since this correspondence is established constructively in the course of an action one is lead to use phrases like: we multiply 4 times 5 to obtain 20 even if there is no genuine operation which manipulates any objects (like number symbols). One can not transform the mathe-

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matical objects (considered as mental entities) in any way but only their representations by symbols. The "operation" on or with the mathematical objects is just the putting them in relations with one another which is induced by actions on prototypical materializations or on symbolizations. We term this usage of verbs as verbal operationalization, the action then occurs only in the speech as a speech act.

Example: Rotation in the plane. In the previous example I have exhibited the features which I judge to characterize constructive abstraction. This example will demonstrate the applicability of this theoretical model in a geometric context. The action here can be: rotating a wheel, the hands of a watch and the like. The relevant mathematical objects are then circle and directions (in the euclidean plane). The mathematical circle cannot be rotated like its materializations. Focus of the attention on directions (e.g. associated with the wheel) will construct a correspondence between initial directions and final directions in the course of material rotations, expressible as a certain angle . As a speech act this is formulated e.g. as "each direction is rotated by a certain angle". The mathematical operation "rotation" then is the corresponding mapping of the circle onto itself which can be operationalized by matrix multiplication (i.e. by symbolic actions on symbols for the points of the circle). This process perfectly fits the general schema depicted above.

Summing up the conclusion here is the following: A mathematical operation does not schematize "corresponding" material actions (which might not exists at all) but it is induced by changes of mathematical objects which are associated with stages and/or elements of the actions. A mathematical operation is originally static correspondence which is operationalized by the use of verbs (as speech acts) or by algorithms on symbols for the mathematical objects. The meaning of the operation comprises all actions (and algorithms) which induce correspondences "isomorphic" to the operation.

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ON VISUAL VERSUS ANALYTICAL THINKING IN MATHEMATICS

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At recent PME conferences and in the literature in general researchers have been strongly advocating the need for special programs to help children and students develop their spatial visualization abilities (e.g., Ben Chaim et al., 1985; Gaulin, 1985). Underlying this plea is the tacit assumption that spatial visualization abilities are necessary, although perhaps not sufficient, for success in higher level mathematics. Clement (1983) stated that "The ability to perform relevant spatial transformations, ... (is a) crucial skill for solving non-trivial problems." Often, statements such as this are interpreted inversely: If one doesn't have spatial ability, then one can't succeed in higher level mathematics.

The Peter principle of management states that a person rises to his or her level of incompetence. It seems as though an analogue of this principle operates in mathematics learning: Namely, one continues to take mathematics courses until one doesn't succeed any more. In the United States enrollment patterns show two conspicuous drop-off points (Stanley et al., 1974): High school geometry and multivariable calculus. Both these courses are highly visual in nature, and both seem to have high failure rates. Is this so because the students can't handle the "relevant spatial transformations"?

It is certainly true that spatial visualization plays a role in mathematical thinking in general, and in concept acquisition and problem solving in particular. But the influence of spatial visualization abilities may well be highly overrated. In calculus, for example, spatial visualization is commonly used for explaining the main concepts, the derivative and the integral; also, it often provides the motivation for the development of algorithms. But the demand placed on the student for actively applying spatial abilities is usually minimal. How many students or experts, when faced with finding the tangent line to $y=x^2$ at (1,1) or when using Newton's method, will accompany their solutions by a drawing? More generally, spatial visualization abilities are usually important for a first encounter with a concept. But after the concepts have been internalized by the students and the algorithms have been practiced, the underlying spatial motivations are often completely abandoned.

Visual thinking might be less natural than we assume (Shell Centre, 1984, p. 18). Indeed, for many students visual thinking and analytical thinking seem to be dichotomous modes, with the analytical mode being overwhelmingly stronger, as measured by the frequency of use. This seems to be happening even though we fight against it in our lessons and recent textbooks are using far more graphics than they have in the past. Does visual thinking come naturally to us as teachers of mathematics? Do experts themselves divorce graphical thinking from analytical thinking and tend toward the latter if given the choice? This question is not new; mathematical reasoning patterns have been studied for many years: Hadamard (1945) has classified types of mathematical minds, and already Poincare (1902, 1904) has contrasted Riemann, a completely visual thinker, and Weierstrass, a completely non-visual one.

Rationale

The hypothesis of this study is that analytical thinking overrides visual thinking, even in experts in mathematics. This hypothesis arises from several different sources, most of which relate to mathematically gifted students. Clements (1984) in interviewing the mathematically precocious Terence Tao whose spatial abilities are exceptionally well developed, observed that Terence preferred to use analytical methods whenever they occurred to him, even when this required more complicated thinking than the visual methods which could have been used instead. In another study, Lean and Clements (1981) concluded that "spatial ability and knowledge of spatial conventions had less influence on mathematical performance than could have been expected from recent relevant literature." Similarly, Krutetskii (1976, p.351) observed that neither an ability for spatial concepts nor an ability to visualize abstract mathematical relationships are obligatory in the structure of mathematical giftedness. Burden and Coulsen (1981) even showed that persons who prefer analytical to visual methods tended to perform better on spatial tests! Hence, several researchers are saying the same thing: The ability to visualize may not be as crucial a requisite for success in higher mathematics as we once thought.

According to the literature, less is known about the role of spatial visualization in average ability students. In an introductory university level course taught by one of the authors (T.E.) *every* inequality problem (more than 30) was solved both graphically and analytically. The graphical method (Dreyfus and Eisenberg, 1985) was stressed, with its advantages over the analytical method being pointed out in each problem. But on the exam less than 5% of the students ($n=97$) opted for the graphical solution.

Experiment

To explore the above hypothesis, questions were constructed which could be solved by both a visual and an analytical approach. The solution paths were equally likely in the sense of one not being more obvious or sophisticated than the other. After a pilot test, some questions were dropped, others revised; eight questions were retained, among them:

1. Is there a quadratic function whose graph passes through the points $(-1,2)$, $(1,-1)$, $(2,3)$, and $(5,1)$?
2. Solve $|2x+6| > 3$.
3. Find the area enclosed between the graphs of $y = x^3+5x^2+9x+8$ and $y = 3x+8$.
6. The vectors $v=(1,0,-1)$, $u=(-2,1,3)$ and $w=(-3,2,5)$ are linearly dependent. What does this mean to you?
8. Among 280 students who were accepted into the Faculty of Science, 160 had taken mathematics in high school, 130 had taken physics and 160 chemistry. 50 had taken mathematics and physics but not chemistry, 20 had taken mathematics and chemistry but not physics, and 30 had taken all three subjects. How many of the students had taken physics and chemistry but not mathematics?

These questions were given in either interview or paper and pencil form to three groups of "experts" in mathematics. These groups were research mathematicians (RM; n=6), high-school mathematics teachers (MT; n=6), and third year university students (ST; n=6). A two minute limit was placed on each problem and it was repeatedly emphasized to the subjects that we weren't interested whether or not they solved the problem but rather in their method of attack. They were encouraged to think aloud where appropriate.

Findings

The response of each subject to each question was classified as being analytical (A), visual (V) or mixed (M). The columns labelled 1 through 8 in the following table report this classification. Bars (--) indicate that the subject didn't or couldn't relate to the question. The first column in the table simply numbers the subjects within their groups for easy reference, while the last column contains the subjects' self-classification (S-C) in response to the question: "Are you a visual thinker?". The answers to this question were classified as yes (Y), no (N) or half-and-half (H).

Classification of responses

	1	2	3	4	5	6	7	8	S-C
RM1	A	A	V	V	A	V	A	A	H
RM2	A	A	A	--	A	A	A	A	N
RM3	V	A	V	V	V	V	V	V	Y
RM4	A	A	V	M	A	A	V	V	H
RM5	A	V	V	V	A	V	V	V	H
RM6	A	V	V	V	V	V	V	V	H
MT1	A	A	--	V	V	--	A	V	N
MT2	A	M	A	A	A	A	A	V	N
MT3	A	A	V	A	A	A	A	V	Y
MT4	V	V	V	V	A	M	V	V	Y
MT5	V	M	V	A	V	M	M	V	Y
MT6	A	--	V	A	A	A	A	V	N
ST1	A	A	A	A	--	A	V	V	H
ST2	A	A	M	A	A	A	V	A	Y
ST3	V	A	A	A	A	A	A	--	Y
ST4	V	A	A	V	V	A	A	A	H
ST5	A	A	A	M	A	A	A	A	Y
ST6	A	A	A	A	A	A	A	A	H

Even a cursory review of this table indicates several striking results:

a. Within the limits of the problems chosen for this investigation, visualizing appears to be rather independent of the problem type - each problem provoked at least 20% visual and at least 30% analytical answers.

b. There are visual subjects (RM3, RM5, RM6, MT4) and analytical subjects (RM2, MT2, MT3, ST2, ST3, ST5, ST6). Although the research mathematicians were more visual than the students in our sample they were far from being completely visual. In fact, RM2 was completely analytical.

c. If any conclusion may be drawn from such a small sample, it is that most research mathematicians and teachers were somewhat more flexible than the students in choosing their approach to a problem.

d. Self-classification was accurate among researchers and teachers (with the exception of MT3 and possibly RM6 and MT1), while the students had a tendency to classify themselves as visual thinkers even after consistently taking an analytical approach.

Discussion and Implications

The interviewees often made interesting side remarks. One from the RM group claimed that a majority of women in mathematics are algebraists and therefore non-visual thinkers. This observation is certainly valid with respect to RM2, a woman algebraist who repeatedly commented during the interview about her inability to visualize mathematical structures. To a lesser degree, the observation also applies to MT2, another woman who, however, attributed her analytical way of thinking to her education. Several other subjects (RM6, MT3) also attributed their mode of thinking to the mathematical education they received. Others stated they had reeducated themselves consciously and were now thinking much more visually than they used to when they were in college. RM3, on the other hand, explained his tendency toward visual thinking by his slowness in thinking about mathematics in general. Several interviewees commented on the relation between their familiarity with the topic of a problem and the approach they chose; e.g., "This is not intuitive to me, so I can't do it graphically." or "I don't know what to do here; therefore I would first draw." It is interesting that these comments went both ways, in one case even within one subject.

With respect to the curriculum, the results of this study favor a parallel development of topics that are visual in nature and topics that are analytical in nature, as opposed to a sequential approach such as is common in the United States. They also favor developing every topic with its analytical as well as with its visual aspects, thus allowing each student to grasp the material in the way which is closer to his cognitive orientation. Possessing or not possessing visual thinking abilities should not be used as the acid test for barring students from studying higher level mathematics.

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THE STEP TO FORMALISATION

KATHLEEN HART

The research project "Children's Mathematical Frameworks" (CMF) was financed by the ESRC at Chelsea College during the years 1983-85. It followed and built upon the research projects CSMS and SESM (previously reported at PME). In the first, a description of levels of understanding in 10 topics commonly taught in the secondary school was formulated from the data obtained from both interviews and more formal testing. The test results showed that many pupils committed the same error when attempting certain questions. The reasons for some of these specific errors were investigated in the subsequent research (SESM). These misconceptions do not begin at the secondary school stage but are formed earlier. A possible learning experience during which such misconceptions can arise is when the child is required to move from a practical or material based approach to mathematics to the formal and symbolic mathematical language used in the secondary school. CMF was designed to monitor this transition to formalisation in a number of topics taught to children aged 8-13 years.

The investigation was based on some classroom observation but mainly on the results of interviews with children. The research methodology involved:-

i. the recruitment of volunteer teachers;
ii. the writing of a scheme of work or teaching plan, in some detail, by the teacher. These were discussed with the researchers who did not seek to influence the way the teachers taught;
iii. the interviewing of six children in the class to be taught, before the specific teaching of a topic took place;
iv. the reporting by the teacher of any changes to the lesson plan and the provision to the researcher of work cards and assignment sheets etc used in class;
v. the interviewing of the six children immediately before the formalisation experience was planned to take place;
vi. the observation and tape recording of the formalisation lesson(s);
vii. the interviewing of the six children immediately after the lesson(s);
viii. the interviewing of the same children three months later;

ix. the transcribing of all the tape recordings;
x. the discussion and analysis of the transcripts.

British primary school teachers have been encouraged, for at least the last 20 years, to use a "concrete" approach to the teaching of mathematics. Their training has emphasised that young children are able to work with materials when they may be unable to appreciate abstract statements (a philosophy loosely based on the theory of Piaget). Over time this emphasis has changed into a series of beliefs which include (a) formal mathematical statements can be seen as generalisations made from a series of practical experiences eg the child recognises the formula for the area of a rectangle from a tabulation of results obtained by building up rows of tiles to fill a number of rectangular spaces (b) if a child moves on to a formalisation from practical experiences he can easily move back to the use of concrete materials if need be (c) children who have difficulty with remembering the symbolism can be told with advantage to revert to materials eg "Use the bricks to help you". The CMF data give valuable information on the truth of these beliefs.

The sample for CMF was composed as follows:-

Area Formula	3 classes aged 9-10 years
Equations	3 classes aged 11-13
Enlargement	4 classes aged 10-12
Volume of a Cuboid	4 classes aged 10-12
Circumference of a Circle	2 classes aged 11-12
Equivalent Fractions	3 classes aged 10-12
Subtraction and Place Value	4 classes aged 8-9 and 1 remedial group aged 12.

Each teacher decided on the content and length of time for the teaching sequence leading up to the formalisation and then designated the lesson in which the formalisation/generalisation or symbolic form would be introduced. It was this lesson that was observed and tape recorded by the research team. In some cases the teacher gave the children material and symbols to work with side by side, on other occasions the material was mentioned as having been used in the past (eg last week) but the lesson itself concerned symbolisation. In one case the teacher provided an intermediate stage between the use of bricks and an algorithm when he made "strokes" on the board to represent the bricks, but this intermediate step was not positively recommended to the children.

The subtraction algorithm in each of the four research classes was based on decomposition and part of the teaching sequence was devoted to the relationship between tens and ones. Prior to the teaching, four out of the seventeen 8-9 year olds interviewed regarding a written subtraction question removed the tens value first. Later, children were observed to do the same when using Dienes' multibase and Unifix. With Unifix, 56 - 28 resulted in the removal of three tens from five tens and then the addition of the "extra 2" to the "6". This is a straightforward and sensible use of materials to solve the problem but it does not mirror the algorithm which is seen as the generalisation arising from these concrete experiences.

The formula for the volume of a cuboid, V = l x b x h was taught to four groups and usually the stage before the triple multiplication was explained by the teacher as "a layer times the number of layers". The area formula was seldom mentioned and children often found "the layer" by counting. In 13 of the 20 cases where a correct general method was remembered three months after the teaching, it was recalled in the form of "layer times number of layers". An additional problem faced by children who remembered that they were expected to multiply was that they had no method for multiplying three numbers. The lessons we might learn from these examples are that (a) the method the child uses with materials is not necessarily that which is generalised in the formula or rule and (b) some generalisations are more successfully assimilated than others as in the case of "the number of layers".

Sometimes the teachers preferred to teach a group of children whom they regarded as being at about the same level of readiness for the idea being presented. Figures 1 and 2 show the performances on subtraction of children aged 8-9 and taught by the same teacher. The axis across the page is marked to represent the four interviews and the other axis is marked with typical responses given by the children (they are not ranked in order of worth although the incorrect methods are placed lower). Note that it is possible for a child to have a good workable method (such as "counting-on") prior to the teaching and to replace it with an incorrect method like "always subtract the smaller digit".

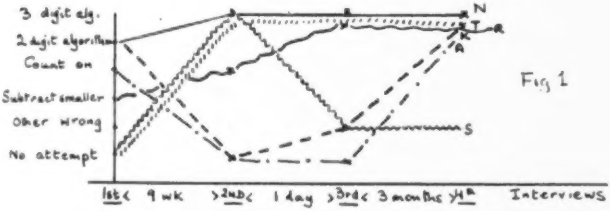


Fig 1

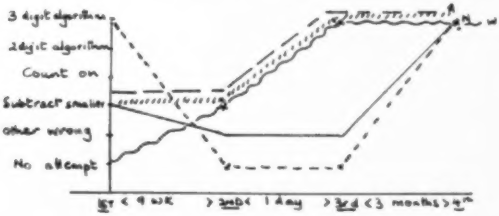


Fig 2

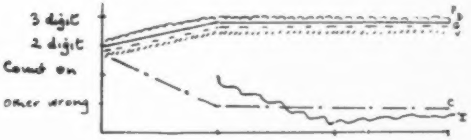
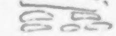


Fig 3

Figure 3 shows the performance of a group of 12 year olds who were regarded as in need of extra help. This response pattern more closely resembles the results expected by teachers, than those shown for the younger first time learners.

The four formalisations discussed here involve operations on numbers and for completion addition or multiplications bonds are used. Many of the children observed trying to use these formalisations had neither of these sets of facts available and so were forced to count. The rules however were not generalisations of counting and the effort expended by the children was on the mechanics of finding the correct number rather than interpreting the rule.

Children in the two classes which were taught the formula for the circumference of a circle retained both the symbolic form and a practical method for finding the distance round a cylinder. One being used when the radius was given and the other when it was difficult to find. In most other cases the practical methods were not remembered. The 10-11 year olds who were learning the rule for generating equivalent fractions found it particularly hard to provide a concrete model when they forgot (or did not understand) the rule. One teacher had used subsets of a set as well as a region model and provided evidence of the reliability of the rule by appealing to fractions with denominators which were factors of 12. The other two teachers had drawn (freehand) a circle, or pair of circles, and called it "a pie" as they subdivided it. The inaccurate division of circles does not show that $3/7 = 9/21$ although if you already know that fact it might boost your confidence. If you do not know the fact and you are trying to find through the use of a set of discs, an equivalent fraction, then you do not know how many discs to take as an initial set and so might reason as Terence did:-



Terence: I'm breaking them up, like we did in class.
Interviewer: I see. They're supposed to be bricks.
Terence: Yes, measuring up .... tenths. How many? ....
Interviewer: 6/10. I asked was there another name you could give to 6/10, that fraction.
Terence: It's more than half.

LATER b)

Interviewer: Now, in this list here, I've got a whole series of fractions. Now, are there any of them worth the same? Could you write down any which are worth the same? 3/8, 6/12, 6/11, 6/16, 10/26, 9/2 .
Terence: draws (b).
Interviewer: Tell me what you are doing. You've done a whole set of little circles there.
Terence: Yes, 12.
Interviewer: There's 12 of them, yes.
Terence: So .. er.. 12, 6. They would be up to there, see (marks halfway) six twelfths.
Interviewer: You've drawn 12 circles.
Terence: I marked in 6/12. Three eighths, 1, 2, 3, 4, 5, 6, 7, 8. So the whole one would be there oooooooo oooo and 3 would be in between. These two are not the same.
Interviewer: They're not the same. Right. 3/8 and 6/12 ... we've knocked out because ... Now what did you do? I have to tell the machine what you did. You put your hand over the last four of those circles.
Terence: Yes, because there's 12 here, but if I put my ... hand over there to show that was 8 and 3/8 would be there and it isn't the same as that (pointing to 6 circles). Try 6/11, is the same (counts from LHs of circles and gets to same point).

To expect a child to invent the concrete model of an abstraction he does not understand is usually too difficult a task. In most cases the children do not remember the physical embodiments after the formalistion lesson has taken place and so cannot refer back to them.

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Meta-cognition: the Role of " Inner Teacher "

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1. Fundamental Conception and Methodology of Research

Recently, meta-cognition has come to be noticed as an important function of human cognitive activities among researchers of mathematics education as well as among professional psychologists. But even the definition of meta-cognition is not yet so firmly settled, and results from the researches have little implication useful to the practice of mathematics education, only still remaining at the psychological interest.

In our research, meta-cognition is defined in the teaching-learning context so as to be applicable to the practice of mathematics education and this is, we believe, the first point in which our research is characterized as being different from others.

We started from a very primitive view that teaching is a scene where a teacher teaches a pupil and a pupil learns from a teacher. But in the process of teaching, a phenomenon which is very much remarkable from psychological point of view will soon happen in pupil's mind; that is what we called *the splitting of ego* in the pupil, or we might call it *decentralization* in a pupil, if we use the Piagetian terminology.

Children, as Piaget said, are ego-centric in their nature, but perhaps as early as in the lower grades of primary school, their egocentrism will gradually collapse and split into two egos: the one is an *acting ego* and the other is an *executive ego* which monitors the former and is regarded as the subject of meta-cognition. Our original conception is that this executive ego is really a substitute or a copy of the teacher from whom the pupil learned. The teacher, if he/she is a good teacher, should ultimately turn over some essential parts of his/her role to the executive ego of the pupil, and after then in a mean time he/she should disappear before the pupil, and in this meaning we refer to the executive ego or the subject of meta-cognition as *inner teacher*.

The second point that characterize our conception, is that we come into the possession of a new methodology to study meta-cognition. Until now, introspection has been almost the sole and often unreliable method of study in this area. But in our case, if we study meta-cognition, we only analyze actual behaviors of the teacher in his/her teaching and closely observe what part of his/her behaviors could be transformed to a pupil as the quality of pupil's meta-cognition.

Thus, we could study meta-cognition through analyzing the process of lessons and could do it in close connection with the practice of mathematics education.

2. The Aim of Research

Ego-splitting is regarded as a phenomenon in the development of a child, but it is also a thinking phenomenon which can be seen in every place of learning. Therefore, the study of meta-cognition can be approached from two points of view: the one is development-psychological which needs a longitudinal observation, and the other is thinking-psychological and this latter view-point is our main concern in our study.

From thinking-psychological view-point, the importance of meta-cognition exists in its activiating function of knowledge to warrant the learning mathematics in its genuine sense. Our aim is to be aware of meta-cognitions through observing lessons and, if possible, to categolize them along with the teaching process. And, we hope, the results of our research would be able to answer to one of the major problems of mathematics education suggested by Dr. Freudenthal(1): *why Johnny can't learn mathematics.*

3. Observation of Classroom Lesson Process and Considerations

Most classroom lesson proceeds through some stages as follows:

1) introduction, 2) development, 3) exercise and application,
4) conclusion.

In 1), teacher presents some topic in various kinds which includes mathematics to be learned in the lesson, and talking about it he/she gradually concentrates his/her pupils' attention to the mathematical core of the topic.

The stage 2) is the main part of the process. Refering to the introduced topic, teacher formulates the content to be learned into a problem or elaborates it as a concept, and ultimately he/she makes pupils learn the method that is used and the concept that is formed as mathematics.

Through the stage 3), teacher makes pupils get possession of the knowledge and skill more firmly.

The stage 4) is the final confirmation of what is learned in the lesson.

Here we confine our consideration to the first two stages, because other two seems not to have much importance to our thesis.

In general, classroom lesson can be looked from two different view-points: one is as a cognitive process of an individual pupil and the other is as a social or group process of the whole classroom. These two are dependent to each other and we never make light of the importance of the latter which is highly emphasized by Dr. Bishop(2) in his recent papers. But here we will be mainly concerned about the former process, because this is the better place to observe the traditional and even current classroom lesson and to illustrate our thesis in this paper.

At first, we should discern two kind of contents that are treated in the lesson: *mathematical* and *material*. For example, when we are to teach the addition of the whole number $2+3=5$, it is mathematical content, but to teach it to the 1-st graders, we use some story or situation which includes this mathematical content. We refer to this story or situation as material content.

In the following we will take notice of a teacher's behaviors and utterances in each stage of a lesson and consider how they could affect on the ways of thinking of an individual pupil. When a pupil solves a problem or use mathematics by him/herself, he/she does not only use the mathematical contents that he/she learned in school, but also needs the 'knowledge on mathematical knowledge' that is 'meta-knowledge'. And it is our aim to illustrate that most of these 'meta-knowledge' comes from his/her teacher's behaviors and utterances.

1) From Introduction-Stage

We should notice on the didactical roles of 'material content' that was refered to in the above. Those roles will be mentioned as follows:

(1) The material contents confines the pupils' awareness to the fixed topic which includes the mathematical contents to be learned.

(2) It will be used as the 'learning-aids' for pupils to understand the mathematical contents.

(3) It will become a model of applying mathematics to other problems or in other situations.

The material content is not purely mathematical, but is essential to learn or to use mathematics. So we may call it a kind of 'meta-knowledge'. We should notice to the fact that it is selected and adopted by the teacher and used by him/her in the lesson, and that it necessarily reflects the whole personality of the teacher more or less, and the more effective his/her lesson is, the more strongly it will control the ways of thinking of the pupil that he/she taught. In reality, in telling a story and presenting a situation, teacher transforms his/her behaviors to pupils, including ways of his/her talking, writing and thinking, and its effect will be the more if pupils are younger.

In this stage of the lesson, a teacher often says as follows:

"Let us consider ...", "Let us try to do ...", "Will you like to do ...", "I wish you to do ...".

Perhaps these may show Mr. Brousseau's(3) 'didactical contract' but we fear that these contracts are often made from the teacher's side exclusively, and this would be the main reason why autonomous or voluntary behaviors of children are scarcely expected in their future lives from mathematical points of view.

After pupils are introduced to the situation that is proposed by the teacher, he/she indicates the problem or conflict imbedded in the situation and makes pupil think how to resolve the contradiction, and as a tool of solution he/she often introduces a new mathematical concept.

For examples, in the 5-th grade, teacher begins the teaching of multiplication by decimal fraction by presenting a verbal problem like this:

"What is the weight of oil *0.6l* if *1l*is *875g*?".

And discussion follows, making pupil be aware of the fact that they can no more understand multiplication as an iterated addition.

In this stage, the frame of teacher's utterances is rather narrow:

"How will you do, if ...?", "Can you do anything?"

But the point is whether the problem itself becomes the pupil's possession or it remains as teacher's. How is it possible that the teacher's problem is transfered to children and becomes their own? We think it depends on the teaching ability of teacher but it rather depends on the acting ability of teacher as Dr.Polya(4) says that teacher should be an actor or actress. This means that the emotional impression would be one of the essential component of meta-cognition and of 'inner teacher' who indicate pupil what to do voluntarily. Indeed, the pleasure of curiosity, excitement of exploration and fascinating imagination would be essential qualities of meta-cognition in learning mathematics and these qualities should be carefully cultivated through all stages of the lesson.

2) From Development-Stage

In most lessons, this is the stage of teacher's explanation and discussion among pupils. We do not neglect the importance of the discussion among pupils, but here we will concern about teacher's explanation, because it seems in our country that teachers like explanation rather than spending much time for the discussion among pupils.

These explanations are carried on such a way of talking as follows:

"It follows that ...", "It is because ...".

Of course a teacher often asks some questions to pupils to confirm or evaluate their understanding. In such a case, he says:

"That's right.", "Is it true?", "It seems not to be ..."

Clearly the teacher wants to urge pupils' reflection and make them examine the answer by themselves, but it is important that pupils become to be able to reflect or examine of their own accords.

In connection with the formation of meta-cognition in this stage, we should like to mention some noticeable facts in this stage:

(1) In explanation, both teacher and pupils resort their reasoning or justification to two things: one is logico-mathematical knowledges which are supposed they have in their intellectual stage, and the other is the material contents that are introduced at the beginning of the lesson. Naturally in lower grades, the latter is more often used than the other and the latter often constitute 'meta-knowledge' of mathematics. We often neglect the latter, but they are as essential as the pure logico-mathematical contents for pupils to learn or use mathematics. In addition, we should also notice that these two categories are closely connected to each other. For instance,

$$876g \times 0.6(l) = 87.6g \times 6(dl) = 87.6g + \dots\dots\dots + 87.6g$$

is not purely mathematical understanding; pupils understand this in resorting to the material knowledge.

(2) Psychologically interesting is that even in a personal thinking there are two subject of thinking as are in this stage of classroom lesson: one proposes a tentative answer and the other asks its justification, or one asks a question and the other answers. In a personal thinking, both roles should be played by a single personality, while in classroom one is played by teacher and

the other is by an individual pupil. And which of these two takes the precedence of this process---this is the most critical point. Of course in classroom the teacher takes, but how is it possible for him to do so? It goes without saying that teacher knows everything about the problem situation, but among others the following knowledges would be most related to the questions teacher asks:

What is the problem all about?
What is the essential point of the problem?
What connection the problem has to the knowledge that is already in one's own possession?

These knowledges are different from the mathematical ones to be learned; they are knowledges about the value of the mathematical knowledges and their connection to oneself, and without these knowledges, one can't develop his/her thinking by oneself even in the same situation where one learned. We may say that he/she needs *a teacher in him/herself* who proposes an appropriate question and properly examines the answer to it. This *inner teacher* would be a copy of his/her teacher in school. This means that the ways of questions and that of evaluations made by the real teacher will be come meta-knowledges of the pupil, for better or worse.

4. Concluding Remarks

We are often inclined to emphasis only the pure mathematical knowledge in its education, but in order to make them activate in pupils, we should notice another kind of knowledge, that is 'meta-knowledge' of mathematics. We argued that this comes from the teacher's behaviors and utterances in the classroom or even from his/her whole personality. To collect these knowledges and arrange them into some categories is our aim, but this aim is not completely attained in this paper and we eagerly wish to continue this research furthermore.

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An essay on the epistemology of learning : a longitudinal analysis of the progessive growth of competence in children doing Logo

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Since the very first of Papert's reports on Logo (Papert 1972, 1979) there has been a strong belief by many educators and researchers that learning to program in Logo may greatly enhance the development of thinking. However, a great deal of the data offered up to now to support such a belief are generally held by hard-core researchers as anecdoctical (Krasnor, Mitterer 1984); indeed, even if anecdoctical data can be judged interesting in themselves, being often powerfully illustrative and at times quite convincing, they do not yet have the power to demonstrate the real cognitive effects of Logo. In an effort to be more rigorous toward such a demonstration , Pea and Kurland 1983 borrowed from the classical trend of assessing learning by measuring the level of generalisation of the "presumed" acquired knowledge. But as underlined in Papert's criticism (Papert 1984), the presence of serious methodological flaws in this approach leads more to an impass than to a solution. Furthermore, it seems that until we have gathered data on what and how the child learns while working in Logo, the generalisation approach will be quite hazardous.

In the search for the kind of specific knowledge acquired by a child while working in a given Logo environment, it may be interesting to investigate the different competencies manifested and developped all along the working sessions. Competencies is mainly defined here by the "savoir-faire" dimension, the role of own's action in the learning process (the "how to do", the "knowing how to do" borrowed from Ken Low's theory of competence (1983)). In the Logo context, we presume that the child's actions while interacting with the turtle follow a logic of its own: the logic of the actions characterising the way the actions are organised and linked one to the other. It is presumed too that, through successive experiences , the logic underneath the child's actions will evolved with the variety and level of sophistication of the actions made. So a rigorous and long term analysis of the observed actions of the children in their day to day learning experience in Logo should allow revealing the epistemology of their learning processes. In a way, this quest concerning the epistemology of the childrens' learning process cuts across Papert's notion of fragmented knowledge (Papert 1984). Indeed it calls for gathering data on how the child progressively modifies, through his work and experience, his initial pieces of knowledge or his initial competence and how he progressively "picks up some of the pieces and puts them together" (Papert 84, p. 12), changing at the same time the nature of his competence.

The research paradigm used by Lawler (1981) offers an interesting alternative for the study of how Logo may affect the development of competencies in children. Lawler rigorously traces the progressive construction of knowledge of a child through a long term analysis of all the competencies and skills effectively shown while working on a specific subjet matter in various real-life microworlds. Even if the present research context does not allow following the observed children in different microworlds, we still have a great deal of data from which we can analyse the course of development of one, two or more competencies in the single turtle geometry microworld. It seems to us, for example, that a long term chronological analysis of all the different uses of important notions inherent to Logo (procedural thinking, iteration, variable, and so on) by different target children, may allow tracing the evolution of competencies and thereby facilitate the identification of what was effectively learned by each child and how it was learned.

The research presently undertaken aims at analysing the course of development of two main competencies in Logo: the ability to program in a procedural way (linked to the development of procedural thinking) and the ability to work with the repeat command (linked to the concepts of iteration and multiplication - c.f. Kayler 86). In this paper we shall explore the various types of progress made by different children in their use and organisation of the procedural thinking through their year long contact with Logo. In an other paper (Lemerise 86), a similar analysis is undertaken but in this case for the repeat command.

We are now in the second year of following a group of 21 students aged between nine and twelve years old, who come once a week to a laboratory equiped with eight computers. Children work either in pairs or alone for approximately an hour each week. Most of the time children work on a personnal project and, in general, take three to four weeks to achieve what they undertook. The dribble files and the protocols of the observers constitute our main sources of data. Four observers systematicaly follow eight target children during each of their working sessions; the other children are also observed, but on an irregular basis. The Logo environment provided is rich in support (the observers are participant-observers guiding the child when need be) but low in formal instruction.

The protocol analysis (from dribble files, observational records and production's analysis) reveals very interesting progressive steps in the elaboration of the procedural action. We intend to present here a few "developmental cases" each one starting with 1) a description of their base line competency; 2) a summary of their successive types of procedural uses -the way procedures, subprocedures and superprocedures are each time organized -; 3) the observed links between succesive types of use (what events give birth to the new way of using subprocedures, for example); 4) the verbalisations of the child, when available, relative to the present topic, and finally 5) the level of competency reached at the end of the school year.

Even though slight differences may appear according to the age of the child (9,10 or 11 years), the personnal working style or the different nature of the chosen projects, a general pattern of six developmental steps characterizes the path usally followed by our young boys and girls in the progressive building up of procedural competencies.

Level 1: Characterized by the absence of subprocedure. There is one main procedure usually with a long string of commands. The procedure is seen as a saving device and the child's actions are planned "de proche en proche". At the follwing session, the procedure is called back and new commands are added; this process is repeat until the project is finished.

Level 2: Subprocedures appear, but mainly because of a time constraint, the working sesion being over. For exemple, at the end of session 1 the child saves under ROBOT what he had time to do; in session 2 he saves under a "new" procedure ROBOT2 his first part (ROBOT)plus a series of new commands.... and so on until completion of the project. Here the child lives a very first level of experience with the embedding of a procedure in a superprocedure.

Level 3: Now subprocedures correspond to parts of the child's project, but most of the time the parts are either time defined or linguistically defined. As in this exemple, where the child wants to do a submarine. In the first session there is only time to do the front part, so he saves it under SUB. The rear part of the submarine is finished during the second session and saved under MARINE. Then it is easy to think of editing a superprocedure SUBMARINE. In general children love those "words game" and through them exercice a lot of their procedural competencies. Take another child, who in the midst of a session defines R for a part of her robot's leg; then later RO (including R plus other commands) for the right side of the body, and then ROB (beginning with RO) and so on until ROBOT is completed. Here each procedure does not correspond to a real part of the object, what seems to determine the content of each one is more the number of commands involved, so they are more easy to correct if their is a bug ! The next project of this child is a rocket (Fusée, in french), where the procedure F stands for the base , U for the body, S for the head, E for an antenna and because there's a E left a moon is defined. It is of course very easy after that to create a superprocedure FUSEE. And it is those two successive experiences that bring the child to level 4.

Level 4 : Here subprocedures correspond to logical parts of the project and are often embedded one in the other. In general, the main lines of the action are planned in advance, but still a lot of organisation of the action emerges during the process itself. A child plans a bird with a head, a body, some feet. While constructing the contour of the head, she thinks of the parts inside (mouth, nose, eyes); similarly, while doing the body, she thinks of the wings. So at the end she has a superprocedure BIRD calling for HEAD, BODY, WINGS and FEET; then HEAD is composed of three subprocedures MOUTH, NOSE, EYES and WINGS for its part contains RIGHT.WING and LEFT. WING. Of course, here the procedures are not at all exportable, being dependent on one another. Indeed, the interface between procedures is undifferentiated from the other commands, being either at the end or at the beginning of part's procedure.

Level 5 : The subprocedures are logical parts, nicely embedded and some of them are easily exportable and transformable. Frederic had already defined CIRCLE in a previous session. He first exports the procedure to define ZERO in his number project, then he uses it again for the TEN, NINE and EIGHT. In a subsequent alphabet project CIRCLE is again called for to produce the letters b, d and p, but it is quickly transformed in HALF.CIRCLE so the previous letters have a nicer look. Then HALF.CIRCLE is transformed too in two new subprocedures HALF.CIRCLE1 and HALF.CIRCLE2 for the need of specific letters. This fith level is already quite sophisticated for our children, but what is still missing is the ability to cope in an organized way with the interfaces. When Frederic, for exemple, wants to write various names with his alphabet he still has a hard time with the interface between each successive letter.

Level 6 : Here, planning the organisation of the interface is also view as a procedural process. As for instance in the project WIZARD, where each part or the weird man was constucted separately and the interface defined afterwards in a procedural mode. Six MOVE were defined going from MOVE.0 to MOVE.5. Again in a CHECK-BOARD project where subprocedures were create to produce an empty square, a filled square, a line of squares; two subprocedures were specifically made for the trajectories of the turtle between two lines of squares ! Finally a superprocedure for the whole check-board was constructed. So this is the top level of sophistication we observed in our study. It must be mentionne that our children rarely use procedure with variable and being so, they did'nt have the opportunity to tackle what Hillel and Samurcay (1985) defined as procedures composed of generalized subprocedures.

In a way, the levels describe above have a lot in common with what Noss (1985) observed in his exhaustive study. But even if these levels are interesting *per se*, they are even more so when one can see how each child goes through them and what kind of context he/she nedds to be able to do so. This is precisely what the data from the "developmental cases" allow us to see. These data may too by the same occasion answer some of the questions posed by Hillel and Samurcay in their search of what can bring conceptual change in the child's view of procedures.

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THE COMPUTER AS MEDIATOR BETWEEN ANALOG AND DIGITAL THINKING

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Introduction. This talk presents a theoretical model to help explain the vitality of the computer for one fundamental aspect of mathematics education. I start from the premise that in order to gain any significant and useful understanding of a mathematical topic, the student (or professional mathematician!) needs to have both an 'analog' and a 'digital' representations of the subject, as well as the ability to shift frequently and easily between the two. (Eg. Davis & Hersh "The Mathematical Experience", pp. 301-316.) I shall not attempt here a definition of the 'analog vs. digital' terminology. Its meaning can be gleaned from the examples, as well as from the following list of closely related pairs of terms: intuitive vs. analytical (or formal), global vs. local, spatial vs. sequential. (Seeing these terms, one is tempted to assign analog and digital thinking to the right and left hemispheres of the brain, but this would be both irresponsible and unnecessary.)

Examples: In geometry, the analog representation roughly corresponds to what might be termed 'geometrical (or spatial) intuition' - the ability to see in the mind's eye figures and relationships, as well as the ability to mentally perform various operations on them. Digital representations may range from simple verbal descriptions ('the diagonals in the parallelogram ABCD bisect each other') to algebraic representations of points, lines and curves in analytic geometry. In elementary arithmetic, children are given an analog representation of the number system via

Cuisenaire rods. In music (to take a non-mathematical example) written sheet-music is a digital representation of the music we hear and play 'analogically'.

The educational problem. These examples help illustrate the claim that at the end of a successful learning process, the learner should ideally 'own' not only the two types of representations, but the equivalence between them as well. Analog representation is important for the creative aspects of mathematics (making conjectures, solving problem, discovering proofs, etc.), as well as for meaningful learning (giving intuitive meaning to the manipulation of symbols). Digital representation is important in order to check our conjectures more objectively and rigorously, and for certain modes of communication (eg. textbooks, programming).

Traditionally, maths education suffers from 'digital chauvinism', which results in an impoverished mathematics (neglecting analog aspects) on top of an impoverished learning process (neglecting students' analog powers). But even when one comes to recognise the importance of analog thinking, it is not easy to design learning environments which promote this kind of thinking, especially the frequent shift between the two modes to help establish the equivalence between them in the learner's mind. This is precisely where the computer steps in.

The digital<->analog cycle. Consider a child programming a HOUSE with the Logo turtle (or, similarly, shooting arrows to pop balloons in Darts, or building logic machines in Rockie's Boots, etc.) The child starts with an analog representation of the house (a mental or actual picture) and is trying to teach the turtle (ie. the computer) how to draw it. The turtle, however, understands only 'digital' instructions (like FD 100, TRIANGLE, etc), so the child, on her own initiative, starts translating her analog house into a digital one. Let us suppose she runs into the famous 'interface' bug, attempting first the following procedure (Fig. 1):

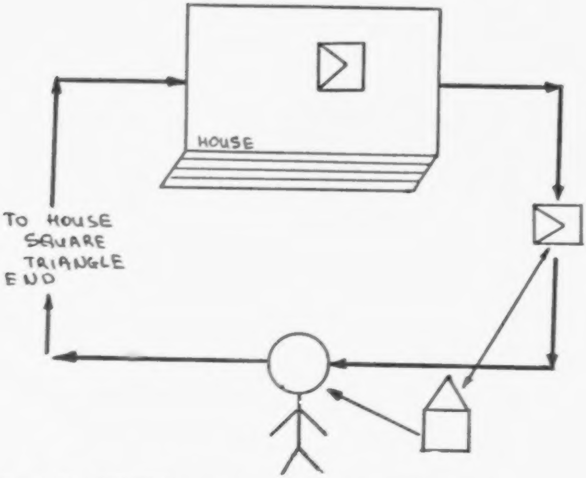
HOUSE
TO HOUSE
SQUARE
TRIANGLE
END

Fig. 1: Converting HOUSE from analog to digital representation

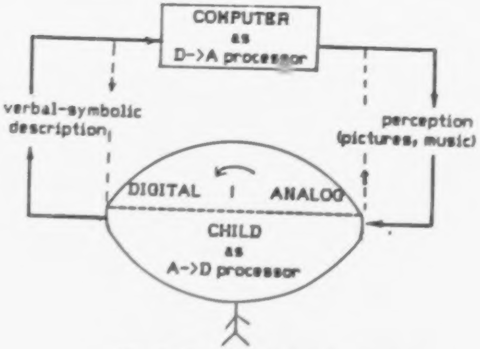
COMPUTER
as
D->A processor
verbal-symbolic
description
perception
(pictures, music)
DIGITAL
ANALOG
CHILD
as
A->D processor

Fig. 2: The digital<->analog cycle.

```
TO HOUSE
 SQUARE
 TRIANGLE
END
```

The turtle, having been instructed to draw the house via the HOUSE procedure, translates this digital description back to an analog one, and draws the buggy picture on the screen. The child is struck by the discrepancy between her original picture and the one produced by her instructions to the turtle (children literally jump here with surprise). Consequently, she becomes involved in a rich digital<->analog comparisons and conversions as she is trying to debug her procedure. Eventually, after many cycles, a digital equivalent of her picture is produced:

```
TO HOUSE
 SQUARE
 FD 50 RT 30
 TRIANGLE
END
```

Let us consider the foregoing interaction from a more abstract perspective. We view the two participants in the dialogue, the child and the computer (or the turtle), as having complementary roles in the action-feedback cycle (Fig. 2). The child inputs analog representations (pictures) of the house and outputs digital representations (symbols in a programming language). The computer inputs digital representations (the child's typed instructions) and outputs analog ones (drawings on the screen). The computer's analog output is input by the child and the child's digital output is input by the computer. The cycle is formed by the child acting as an analog-to-digital processor and the computer as a digital-to-analog processor. During the many cycles of debugging and executing the HOUSE procedure, there is a constant translation between the analog and digital representations of the house, until a perfect match is achieved - the symbols actually produce the desired picture. The child have actually established an equivalence between the two representations.

Remarks: 1. In some educational activities (Teacher: "Here is a turtle-procedure. Can you guess what it will draw?"), the flow in the cycle reverses direction: The children are given a digital representation and are asked to 'play turtle' and find the corresponding analog representation.

2. It can be demonstarted that it is really the presence of both the analog and the digital modes, and the interaction between them, which accounts for the educational power of most good maths-educational software (see the dotted lines in Fig. 2). To see an interaction that is purely in the analog mode, short-circuit the analog<->digital cycle on the analog side (eg. have the child drive the turtle with a joystick) and your valuable software has degenerated into a space game. To see an interaction that is purely in the digital mode, short-circuit the cycle on the digital side (ie. have the computer answer with letters and formulas instead of pictures), and you get the sort of interaction that is typical in standard computerised-book CAI.

Is the computer really necessary? This model accounts for the vitality of a (graphically-equipped) computer in such processes - it really takes the full power of a computer to process alphanumeric characters into pictures. Compared to conventional materials aimed at the same goals (eg. rods, blocks or boards), the computer offers both a quantitative and a qualitative leap. Quantitively, there are simply many more opportunities to create analog repreeentatiopns with the computer than with traditional materials, especially when dealing with dynamic aspects of mathematical phenomena. Consider for example Sprites, Darts and various graphical representations of limit processes in calculus.

Qualitatively, the computer is the only medium that can do digital-to-analog processing, thus enabling the cycle in which the child does the reverse analog-to-digital processing. With Cuisenaire rods, for example, the interaction is purely in the analog mode. The rods themselves do not in any way support the development of the all-important equivalence between the two modes. It requires teacher-assigned excercises, and teacher evaluation of the performance on these excercises, to extract the relationships between rodes and numbers. In Logo, in contrast, the equivalence between the numbers that occur as inputs to FORWARD and the distance traveled by the turtle, is developed naturally and spontaneously through the interaction with the turtle. Beside being more effective, this interaction makes it possible to transfer much of the control of the learning process from teacher to child, and to replace teacher evaluation of the child's work with objective, non-judgemental feedback from the computer.

MASTERY LEARNING AND DIAGNOSTIC TESTING

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Since 1983 the NFER has been undertaking a feasibility study on Graduated Tests for Lower Attainers in Mathematics. The project, which deals with pupils in the 13 to 16 age group, is due to finish at the end of 1986. The work, sponsored by the Department of Education and Science and the Welsh Office, derives from a suggestion in the report of the Cockcroft committee [1] (1982) that consideration be given to investigating the feasibility of such assessments. Under examination are the feasibility issues concerned with the development of a series of criterion referenced graduated tests which demonstrate mastery by the pupils by means of high success rates on mathematical activity of value. In addition, it is felt to be important that the assessments considered should be of value in the diagnostic sense. The content of the assessments is guided by the "Foundation List" of mathematical topics proposed by the Cockcroft committee, and a range of assessment modes, including mental, practical and oral is involved.

A variety of issues need to be considered, and in this paper two related concerns are focussed upon, mastery learning and diagnostic testing. In turn, four separate issues can be derived:

(i) how can performance be described?

(ii) is performance as described by a criterion statement consistent for different contexts and other variables?

(iii) how can 'mastery' be defined?

(iv) what is the relationship between 'mastery' and diagnostic assessment?

(i) How can performance be described?

The identification of practicable criteria is crucial in any criterion referenced system, and the question of practicability needs to take account of the needs of the potential audience. On the one hand criteria need to be sufficiently specific to define the domain accurately enough for appropriate assessments to be constructed, while on the other hand information about a pupil's success on a criterion needs to be useful to a range of audiences. Pupils themselves, parents, and employers have to be considered as well as those directly involved in the mathematical education of the pupil.

The process of analysis of the subject matter in terms of the skills, concepts and strategies required tends to lead to a large number of specific criteria. At this stage of the project's work, it seems likely that proliferation of criteria and high degrees of criterion specificity are features which audiences other than assessment constructors and designers of course materials find unhelpful; the information is at too great a level of detail. The question of breadth of description is clearly one which depends in part on the consistency of performance as well as on the manner in which criteria are derived.

(ii) Is performance as described by a criterion statement consistent for different contexts and other variables?

As an example, the task of reading tabular data produces a range of facility values and the factors shown below are associated with difficulty:

- single or double entry table (one way on two way table).
- search requirements (how easy is it to find the information)
- interpretive requirements (eg. 'When is it hottest' has to be interpreted as 'Which month has the highest temperature').

- Computational requirements ('What is the cost of 2 days bed and breakfast compared with 'How much is bed and breakfast?')

If these factors are related to success rates the picture is as follows for a sample composed of the lowest 40% of attainers in mathematics:

Facility in sample	Single/Double Entry	Task demands		
		Search	Interpretation	Computation
> 90%	Single	low	low	none
80 - 90%	Single	some	low	none
70 - 80%	Double	low	low	none
60 - 70%	Double	low	some	none
20 - 60%	Double	(one or more of search, interpretation demands high or computation required)		

It can be seen that a simple criterion for the task of reading tabular data would involve considerable loss of accuracy, while a set of criteria taking into account these findings would involve a high degree of qualification in terms of conditional statements.

As a further example, if whole number place value is considered, a crucial factor is whether the pupil is required to consider positive whole numbers as complete entities or must consider the column (or place) value of each digit. For the same sample as above, success rates for ordering whole numbers or giving a whole number between two given whole numbers are 95%+, but items on column values produce facilities of 50% to 80%.

The context in which a task is placed also affects success rates and errors made. Several factors have been identified with regard to context, and a central feature appears to be the degree to which a context allows the pupil to use a range of approaches which avoid difficult pieces of mathematics. Contexts where what is mathematically a division can be conceptualised as an addition, subtraction or multiplication are helpful, as are contexts which allow tasks involving decimals to be successfully approached using whole numbers.

Given this rather complex situation of performance being affected, inter alia, by complexity and context, the further question of how mastery can be defined needs to be considered.

(iii) How can mastery be defined?

Two central themes are seen as important; mastery for what purpose and the technical issues which need to be considered following a decision on purpose. The question of mastery for what purpose can be seen in the light of several examples of existing forms of assessment. In a British driving test, incorrect performance of certain manoeuvres leads to failure, but certain faults may be compensated for by good performance elsewhere. In training an airline pilot, no such compensation is allowed. In education, as opposed to training, criteria are generally not so directly applicable to future tasks and the aims are often summarised in terms of basics on which future training might build, or general preparation for tasks likely to be encountered in everyday life. Given this less specific range of aims, it seems likely that in the educational context the working definition of mastery can be, and should be, more flexible and forward looking.

Following this view 100% success is seen as being too strict a requirement, and the project has been looking at the effect of setting different pass marks for items grouped by difficulty level which all test common subject matter - a topic or process for example. The results suggest that in the topics so far examined the pass mark, when varied between around 65% and 85%, produces comparatively little difference in the number of non-scale types (pupils who pass more difficult groups of item having failed easier groups) but does, of course, alter the proportion of pupils classified as being proficient. In any event, the number of pupils having non-scale response patterns has been very small, less than 5%. This suggests that flexibility with regard to pass marks is available.

Flexiblity will, anyway, be required in situations where there is no clear cut method of deciding what a correct response is. The tasks pupils are required to tackle include practical work, problem solving and items on both estimating and approximating. In such cases there is often no clear cut off point for success. The demands for accurate measurement of success will need to be moderated by the need to produce a workable system capable of producing useful information. This concept of useful information is seen not only as covering mastery, but also indicating pupils' difficulties and identifying the sort of error made.

(iv) What is the relationship between 'mastery' and diagnostic assessment?

At first glance there is a conflict between assessments for mastery and diagnostic use. High success rates are implied by mastery, while diagnostic assessment usually derives from incorrect responses. In an ideal system pupils are only entered for a test of mastery when success is highly probable, and failure to achieve mastery, however defined, should be uncommon. The resolution of this potential conflict between mastery and diagnosis is seen as the use of similar assessments during the instruction and learning processes where diagnostic properties can be fully exploited. The presence of the same diagnostic properites in the final assessment to check on mastery is seen as a back up procedure. A range of assessments with diagnostic properties has been developed. An example is the item below which is designed to test decimal place value by looking at the ability of the pupil to place in order decimals representing lengths:

Sarah measured the lengths of some sticks in metres:

		Selected by
Stick A	0.625 metres	14%
Stick B	0.25 metres	4%
Stick C	0.375 metres	0%
Stick D	0.125 metres	4%
Stick E	0.5 metres	74%

Which stick is the smallest? Stick ________

The correct response (D) is very rare, the task is too demanding for current low attainers. What is significant is that nearly three quarters of the pupils tested select (E), a response suggestive of ordering whole numbers rather than decimals. The 14% choosing (A) give responses consistent with the "largest is smallest" error made in such situations. The item is useful in providing guidance as to whether a pupil may hold either of these incorrect views on the nature of decimals.

The paper here has discussed the general aims of mastery and diagnostic testing and more detail will be provided in the research report to be presented.

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[1] The Cockcroft Committee (1982) Mathematics Counts. LONDON: HMSO

STUDENT COGNITIVE ABILITIES AND CURRICULAR COGNITIVE DEMANDS IN ELEMENTARY SCHOOL MEASUREMENT, GEOMETRY, AND GRAPHING.

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The "Assessing Cognitive Levels in Classrooms" (ACLIC) project was carried out during 1983-85 to answer the question: Is there a reasonable fit between the instructional demands implied by the Alberta Elementary Mathematics Curriculum and the cognitive levels demonstrated by students in these mathematical topics? A complete report of the project is available elsewhere (Marchand, Bye, Harrison, & Schroeder, 1985), but the Project's methods and findings with respect to the Measurement and Geometry (including Graphing) strands of the curriculum for Grades 1 to 6 (ages 6-11) will be described in this paper.

Pupil cognitive assessment procedures including individual interviews (for Grades 1 to 3) and paper-and-pencil tests (for Grades 3 to 6) were developed for key spatial concepts. The interview tasks were adapted from the Piagetian literature, while paper-and-pencil tests were assembled mainly from items of CSMS (Hart, 1981) and ACER (Cornish & Wines, 1978). The same criteria used in the making assessments of student responses were applied to four components or aspects of the curriculum: objectives, textbook materials, classroom activities, and Alberta Education Achievement Test items. These produced the assessments of the curricular cognitive demands. Two-sample Kolmogorov-Smirnov (K-S) tests were used to determine whether there were significant differences between the distributions of levels of student cognitive responses and the respective distributions of curricular cognitive demand levels.

Figures 1 to 4 give details of the sample sizes and distributions of levels observed and the results of the K-S tests.

Figure 1: Pupil Response and Curricular Demand Contrasts, Measurement, Grades 1-3

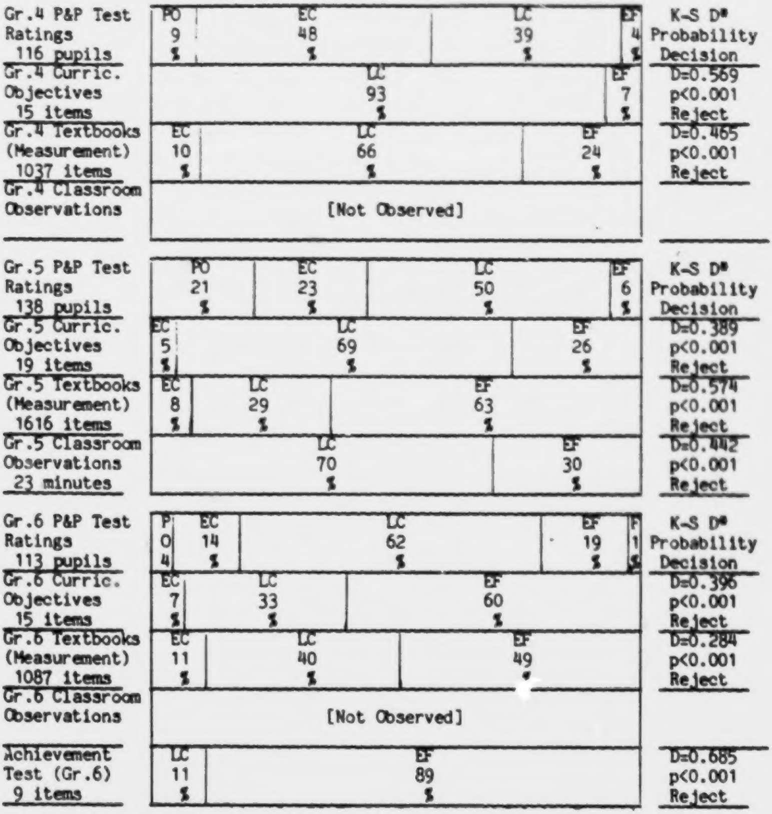
	PO	EC	LC	EF	K-S D* Probability Decision
Gr.1 Interview Ratings 60 pupils	64 %	21 %	15 %		
Gr.1 Curric. Objectives 19 items	26 %	42 %	32 %		D=0.378 p<0.001 Reject
Gr.1 Textbooks (Measurement) 297 items	7 %	81 %	12 %		D=0.574 p<0.001 Reject
Gr.1 Classroom Observations 68 minutes		76 %	2	22 %	D=0.641 p<0.001 Reject

	PO	EC	LC	EF	K-S D* Probability Decision
Gr.2 Interview Ratings 60 pupils	49 %	27 %	22 %	2	
Gr.2 Curric. Objectives 24 items	13 %	29 %	58 %		D=0.369 p<0.001 Reject
Gr.2 Textbooks (Measurement) 753 items	1	43 %	56 %		D=0.484 p<0.001 Reject
Gr.2 Classroom Observations 52 minutes			90 %	10 %	D=0.759 p<0.001 Reject

	PO	EC	LC	EF	Inter-view	K-S D* Prob. Dec'n
Gr.3 Interview Ratings 60 pupils	33 %	37 %	21 %	9 %		
Paper & Pencil Tests (ACER) 92 pupils	27 %	53 %	18 %	2	Paper K-S D* Prob.	0.103 >0.200 Acc.
Gr.3 Curric. Objectives 15 items		13 %	87 %		0.671 <0.001 Rej.	0.568 <0.001 Rej.
Gr.3 Textbooks (Measurement) 722 items		25 %	75 %		0.559 <0.001 Rej.	0.456 <0.001 Rej.
Gr.3 Classroom Observations 40 minutes			75 %	25 %	0.804 <0.001 Rej.	0.702 <0.001 Rej.
Achievement Test (Gr.3) 7 items		14 %	86 %		0.661 <0.001 Rej.	0.559 <0.001 Rej.

PO-Preoperational; EC-Early Concrete; LC-Late Concrete;
EF-Early Formal; F-Formal; *-Kolmogorov-Smirnov Two Sample Test

Figure 2: Pupil Response and Curricular Demand Contrasts, Measurement, Grades 4-6



Gr.4 P&P Test Ratings 116 pupils	PO 9 %	EC 48 %	LC 39 %	EF 4 %		K-S D* Probability Decision
Gr.4 Curric. Objectives 15 items			LC 93 %	EF 7 %		D=0.569 p<0.001 Reject
Gr.4 Textbooks (Measurement) 1037 items		EC 10 %	LC 66 %	EF 24 %		D=0.465 p<0.001 Reject
Gr.4 Classroom Observations	[Not Observed]					
Gr.5 P&P Test Ratings 138 pupils	PO 21 %	EC 23 %	LC 50 %	EF 6 %		K-S D* Probability Decision
Gr.5 Curric. Objectives 19 items		EC 5 %	LC 69 %	EF 26 %		D=0.389 p<0.001 Reject
Gr.5 Textbooks (Measurement) 1616 items		EC 8 %	LC 29 %	EF 63 %		D=0.574 p<0.001 Reject
Gr.5 Classroom Observations 23 minutes			LC 70 %	EF 30 %		D=0.442 p<0.001 Reject
Gr.6 P&P Test Ratings 113 pupils	PO 4	EC 14 %	LC 62 %	EF 19 %	F 1 %	K-S D* Probability Decision
Gr.6 Curric. Objectives 15 items		EC 7 %	LC 33 %	EF 60 %		D=0.396 p<0.001 Reject
Gr.6 Textbooks (Measurement) 1087 items		EC 11 %	LC 40 %	EF 49 %		D=0.284 p<0.001 Reject
Gr.6 Classroom Observations	[Not Observed]					
Achievement Test (Gr.6) 9 items			LC 11 %	EF 89 %		D=0.685 p<0.001 Reject

PO-Preoperational; EC-Early Concrete; LC-Late Concrete;
EF-Early Formal; F-Formal; *-Kolmogorov-Smirnov Two Sample Test

Figure 3: Pupil Response and Curricular Demand Contrasts, Geometry, Grades 1-3

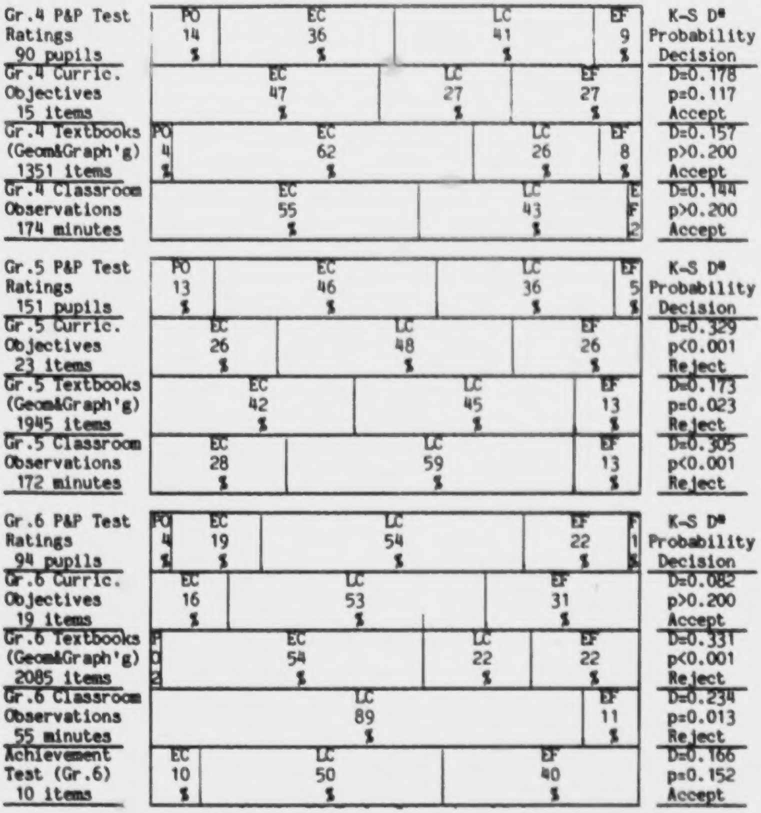
	PO	EC	LC	EF	K-S D* Probability Decision
Gr.1 Interview Ratings 60 pupils	37 %	22 %	41 %		
Gr.1 Curric. Objectives 3 items	33 %	67 %			D=0.413 p<0.001 Reject
Gr.1 Textbooks (Geometry) 251 items	49 %	51 %			D=0.413 p<0.001 Reject
Gr.1 Classroom Observations 14 minutes		64 %		36 %	D=0.367 p=0.001 Reject

	PO	EC	LC	EF	K-S D* Probability Decision
Gr.2 Interview Ratings 60 pupils	23 %	49 %	26 %	2	
Gr.2 Curric. Objectives 7 items		71 %	29 %		D=0.226 p=0.093 Accept
Gr.2 Textbooks (Geometry) 411 items	29 %	55 %	16 %		D=0.129 p>0.200 Accept
Gr.2 Classroom Observations 51 minutes		27 %		73 %	D=0.706 p<0.001 Reject

	PO	EC	LC	EF	Inter-view	K-S D* Prob. Dec'n
Gr.3 Interview Ratings 60 pupils	26 %	40 %	27 %	7 %		
Paper & Pencil Tests (ACER) 112 pupils	24 %	43 %	33 %		Paper K-S D* Prob.	0.072 >0.200 Acc.
Gr.3 Curric. Objectives 12 items		58 %	25 %	17 %	0.241 0.004 Rej.	0.258 0.039 Rej.
Gr.3 Textbooks (Geometry) 1011 items	5 %	77 %	14 %	4 %	0.192 0.035 Rej.	0.209 0.147 Acc.
Gr.3 Classroom Observations 35 minutes		63 %	6 %	31 %	0.314 <0.001 Rej.	0.258 0.039 Rej.
Achievement Test (Gr.3) 8 items		50 %	38 %	12 %	0.241 0.004 Rej.	0.258 0.039 Rej.

PO-Preoperational; EC-Early Concrete; LC-Late Concrete;
EF-Early Formal; F-Formal; *-Kolmogorov-Smirnov Two Sample Test

Figure 4: Pupil Response and Curricular Demand Contrasts, Geometry, Grades 4-6



PO-Preoperational; EC-Early Concrete; LC-Late Concrete;
EF-Early Formal; F-Formal; *-Kolmogorov-Smirnov Two Sample Test

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Considering both the numeric and the spatial strands of the curriculum, it was found that the cognitive demands made by the curriculum and its interpretations corresponded reasonably well with the distributions of student cognitive responses. In most topics and at most grade levels there were some matches, some demand distributions significantly lower, and some demand distributions significantly higher than the corresponding student response distributions. In Geometry (including Graphing) 40% of the demand distributions matched the relevant distribution of student responses, 45% of the demand distributions were significantly higher, and 15% were significantly lower. The best fit was found in Grade 4 Geometry & Graphing where all three of the demand distributions matched the response distribution. Howevr, the findings in the Measurement strand were dramatically different from those of other strands. All of the Measurement demand distributions in all six grades were significantly higher then the corresponding distributions of pupil responses.

The striking mismatch between levels of curricular cognitive demand and levels of students' responses in Measurement suggests a need for materials and teaching approaches that would "bridge the gap" between demands and demonstated cognitive abilities. Further discussion of the Project's findings and recommendations, as well as some recently developed classroom applications of these and similar assessments will be given in the presentation.

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Cognitive Structures of Algorithmic Thinking

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1. Introduction

During the recent years we have studied how children at primary and early secondary level form concepts in the field connected with computer programming. Our interest was to describe and understand fundamental cognitive processes which are running in someone while he is concerned with the invention and analysis of algorithms. Our methodology is to observe single pupils when they are dealing with algorithmic problems presented by the researcher. In this paper we will try to combine 3 aspects of recent research in this area: the role of external representation of a concept, the inner cognitive structure and individual different cognitive strategies. We are dealing with these 3 questions under the special aspect of algorithmic concept formation but we are convinced that our observations and hypothesis are fundamental for a broader area of mathematical thinking.

2. Fundamental ideas

One fundamental idea concerning the understanding of concept formation processes in the field of computer programming is the following: The invention of an algorithm is regarded as the problem of organizing elementary actions, which the computer has to execute, rather than the structuring of the given problem (COHORS-FRESENBORG 1982). In the beginning this idea was the basis of a course to introduce fundamental ideas of computer programming to children at early secondary level (COHORS-FRESENBORG/GRIEP/SCHWANK 1979,1982).

The second idea is concerning the cognitive strategies. In some case-studies with 10 years old pupils which were inventing automata networks with the didactical material Dynamic Mazes (COHORS-FRESENBORG 1978) we found out, that some of the pupils

prefer a strategy, which we classified as **sequential thinking** (SCHWANK 1979). The third idea is concerning the role of **external representation** of an algorithm. Classroom-observations and case-studies had shown that there exist differences, whether an algorithm was invented by playing with match-sticks for organizing actions with natural numbers, or as a computation-network with the Dynamic Mazes, or immediately as a computer-program. An analysis concerning the role of different external representations can be found in COHORS-FRESENBORG (1986).
In SCHWANK (1979) we find the idea that the specific mathematical structure of the Dynamic Mazes, namely that they are sequentially running, may support a specific way of problem solving behaviour: sequential thinking. On the other hand we find the idea, that the three mentioned forms of representing an algorithm form a hierarchie. This was the basis for the lesson courses in the beginning (COHORS-FRESENBORG/GRIEP/SCHWANK 1979, 1982).
The last idea deals with the difference between **inventing** and **analyzing** an algorithm. Even in the beginning of teaching algorithms with the Dynamic Mazes in 1975 we find both types of problems: Inventing a network for a given functioning of an automaton and analyzing a given network by writing down its automaton table (COHORS-FRESENBORG 1978).

3. Experimental research

In several pilot studies there was worked out the following design (COHORS-FRESENBORG 1982,1983): In a situation which may be described as a clinical interview a set of tasks is presented to the pupil by the researcher. The different tasks belong to three categories: in a **constructive** task the pupil has to invent an algorithm, in an **analytic** task it has to analyse a given algorithm and in a **debugging** task it has to analyze a given algorithm which contains an error, and after finding the mistake it has to repare it. If the pupil can not solve the given problem, it receives several hints from the researcher, who has a cataloge of diagnostic points which help him to decide which hint he has to give to the pupil. The set of these hints is the same for all investigated pupils. The number of hints, which has been given by the researcher to a pupil, may be regarded as a measure of its success.

3.1 A first Approach

3.1.1 Pilot Studies

In a first pilot study (COHORS-FRESENBORG 1982) there had been found out that there exist pupils which have very different success in constructing our analyzing algorithms. A deeper analysis of the protocols of those problem solving sessions let to the discovery, that some pupils choose very specific kinds of problem solving behaviour. Some of these pupils started as it could be foreseen: They begin their work on the solution by analyzing the given problem, structuring it and trying to build up a conceptual framework in which they build in their preknowledge about previous problems and their solutions. For this behaviour there was created the terminology **"conceptual"**. Different from this behaviour is the following, so called **"sequential"** which has been mentioned above: Pupils following this strategy are goal-orientated but they start with a first solution before they have completely structured their ideas; they develope their ideas in a dialog with the material; they analyze partial solutions to find the complete solution by modifying them (COHORS-FRESENBORG/KAUNE).

3.2.1 Main Study

After several pilot studies a systematic and sophisticated investigation of these different aspects of algorithmic thinking mentioned above was done by KAUNE (1985). She took one class grade 7 (16 girls, 7 boys, aged 12-14) in a Gymnasium (in this school there are about the upper 25% of the German pupils). Her aim was to investigate the relation between the abilities of the pupils in constructive and analytic tasks, the role of the prefered form of representation of the algorithmic concepts and the prefered cognitive strategy (sequential versus conceptual).

The important results of her studies are the following: She could proof, that there exist pupils with different abilities in constructing or analyzing an algorithm, the external representation describes a world, in which a pupil is thinking, the preference for the match-sticks or the computationel networks of the Dynamic Mazes is for quite a lot of pupils very stable during the problem solving sessions. There exist pupils which have a specific individual preference for one of the two cognitive strategies conceptual (8 pupils) and sequential (8 pupils), but there are 5 pupils which change the strategies. There was the interesting result, that there was no correlation between the sequential cognitive strategy and the preference for working with the sequentially running Dynamic Mazes.

3.2 A second Approach

3.2.1 Special Case-Studies

When I was working with a 14 years old deaf boy I saw, that the action-orientated approach to algorithmic concept formation as it was used by COHORS-FRESENBORG/ STRÜBER (1982) to teach deaf pupils was not successfull with this boy. I could not see, that he was interested and able to organize sequences of actions, but he seemed to me to be very sensitive in reflecting relations between different states and their formal description. I only had success with teaching him after I had decided to resume, that this boy preferes a cognitive structure in which the relations between different mathematical objects and their symbolic descriptions form the basis of his thinking. This hypothesis seemed to me to be very strange because such a thinking in relations is normally expressed in mathematics by the use of predicates and this means by the use of language.

In the second case-study I had the chance to teach a very bright 7 years old boy. Contrary to our previous experiences with primary children he did not like to invent algorithms with Dynamic Mazes. He prefered playing with the match-sticks. When I was analyzing his behaviour I found out, that he always tried to arrange certain relations between f.e. the mathematical objects or between the start situation and the goal. But he did not like and he was not successful when he had to think in terms of functioning of a machine.

3.2.2 Further development of a theoretical framework

The analysis of the above mentioned experimental research led to the following hypothesis (SCHWANK 1985): There exist two different cognitive structures in which the thinking processes are expressed: One structure is built up by **predicates** (relations) and the other one is built up by **functions** (operations) . If we say, that a person preferes predicative versus functional thinking we say that such a person is translating the external given problem into his personal internal conceptual representation: One preferes predicates, the other one functions. The prefered internal cognitive structure must be distinguished from the prefered cognitive strategy in the sense of COHORS-FRESENBORG and KAUNE: **Conceptual** versus **sequential**. This level of cognitive strategy is working on the cognitive structure. By distinguishing this, we now can explain some findings of KAUNE: Some of their pupils with a conceptual strategy prefered the external representation of the functioning networks. Our explaination is that this external representation was matching their prefered internal cognitive structure, the functional one.

3.2.3 First Experimental Tests

The first experimental proof for our hypothesis that our proposed distinction between the predicative/functional cognitive structure and the conceptual/sequential cognitive strategy describes two independent domains of thinking was given by MARPAUNG (1986). In his case-studies with indonesian boys and girls of early secondary level with an experimental design developed from KAUNE (1985) he has found some examples for our hypothesis (see MARPAUNG 1986, p. 85/86).

Our hypothesis may also explain, why some pupils have such difficulties to repare an error in a given network: They do not think in terms of functioning of a machine. Our own now going on research is concerned with weaker pupils (the lowest 35%) in early secondary level. My first impression is the following: Those pupils are more often able to use simple (static) predicates than to understand the functioning of concepts; they are not able to look "into" the mathematical concepts; they prefer a sequential strategy.

4. Prospect

Although the theoretical framework which we have developed here is concerned with the construction and analyzing of algorithms we are convinced, that it is relevant for explaining other fields of (mathematical) thinking. The event, that we have discovered this, being concerned with the explaination of algorithmic thinking, should not only be explained by random: The mathematical field of algorithms is one in which a cognitive structure built up by functions (actions) is a very natural and therefore fruitfull internal representation of the mathematical conceptual framework. But we remind, that outside the world of algorithms there exists a philosophy of mathematics in which description of actions and operations plays an important role.

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The Role of Metacognition in Children's Mathematical Problem Solving

A Report of Research in Progress.

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1. Introduction

Metacognition or 'one's knowledge concerning one's own cognitive processes and products or anything related to them' (Flavell, 1976, p.232), was once the exclusive domain of developmental pschologists interested in metamemory and information-processing models of human behaviour (see reviews by Flavell and Wellman, 1977 and Brown, 1987). Today, it is a focal point of much of the literature on mathematical problem solving (see Garofalo and Lester, 1985). Evidence that children and adults have access to a number of 'out-of-school' strategies for solving problems in context-rich settings (Carraher, Carraher and Schliemann, 1985) suggests a capacity to recognise and monitor one's own cognition as do the expert-novice problem solving protocols described by Schoenfield (1983). The qualitative changes in students ability to think about problems, observed as a result of curriculum projects on mathematical problem solving (see Charles and Lester, 1984 for example), also hint at the role played by metacognition. Although not cited as such, most of the behaviour said to be characteristic of mathematical thinking (Mason, Burton and Stacey, 1982) is essentially metacognitive. The rubric generated processes of specializing, generalising, conjecturing and justifying require cognitive awareness, in that relevant skills and knowledge need to be retrieved, and self-regulation, in that the responsibility for task management rests with the 'thinker'.

Although recognised, the exact nature of the role of metacognition in children's mathematical problem solving remains somewhat of a mystery. This is largely due to the difficulties associated with identifying and assessing metacognitive behaviour, and the paucity of adequate models of mathematical problem solving which accommodate a metacognitive component.

2. Research Questions.

A cognitive - metacognitive model of problem-solving (Lester, 1983), derived from the distinction inherent in Flavell's definition: that metacognition consists of a cognitive self-awareness component and a behaviour regulation component, was used to generate the following research questions:

(i) is metacognition a "driving force" in children's mathematical problem-solving as suggested by Silver (1982) and Schoenfield (1983), and

(ii) to what extent can student's cognitive awareness and ability to regulate their actions be improved through training (Lester, 1983)?

Additional questions to be considered in the study are: the extent to which constructs of this sort can be assessed, what, if any, is the nature of the interaction between the cognitive awareness variables and those concerned with regulating behaviour, and the extent to which affective factors and belief systems interact with the training to qualify/affect performance (Silver, 1982; Schoenfield, 1985)

3. Methodology.

Three grade four classes and one grade six were involved in a ten-week teaching experiment between October and December, 1985. Six children from each grade were interviewed individually after one of the two problem-solving sessions for the week. This session was also video-taped. Before the teaching experiment began, the children selected for interview were asked to solve a two-step problem which was ammenable to the strategy: to work backwards. Teachers were asked to provide information on students at the beginning and at the end of the training sequence and to keep a diary of any noticeable changes in behaviour or reactions to the training experience. At the end of the ten weeks, all the children were tested on their ability to solve a range of mathematical problems and on the extent to which they could report on what they had done and how they had done it. An Attitude Scale was also administered. Where possible this data was also obtained from parallel grades who had not been exposed to the training.

The training involved the regular provision of both written and verbal models of cognitive monitoring and required that these procedures be recognised and engaged in by the children, either individually, in small groups or as part of a class discussion. Different problem types and strategies were considered explicitly and overtly in conjunction with an appropriate model of the problem solving process (see Barry, Booker, Parry and Siemon, 1983).

4. Results

Preliminary findings tend to suggest that the particular form of training did lead to some positive changes in most student's ability to solve problems of a non-routine nature, but that prior knowledge, particularly of a procedural form, and beliefs about the object of school mathematics and oneself as an effective problem-solver seem to play a very important role in determining performance. The following case study illustrates this point.

Julia, a grade 4 student who sees herself as "doing as well as most" in mathematics and whose performance on class tests is generally average, views mathematics as a painful (but necessary) interruption to the things she likes doing. Julia demonstrated a remarkable capacity for retaining very detailed knowledge about what "Mr. Hand did" and used this to solve whatever problems were presented immediately after the teaching sequence. This seemed to be based on her beliefs that mathematics was about "doing something with the numbers to get an answer", that problems presented within the one teaching session were "the same" and that "to be good at mathematics" amounted to remembering and reproducing exactly what the teacher did. Questioned on the importance of meaning, Julia indicated that she did not expect mathematics to have a meaning even though she used labels regularly with confidence.

Interviewed after a session in which a problem involving the modelling or drawing of a fence was required in order to solve the problem, Julia was asked how she would solve a similar problem. She immediately replied: "twelve sevens", asked why she had chosen these (quite inappropriate numbers) she said: "Cos'...ah..we did it this morning". Questioned on what she had to find (cost) and what she had used (length of each piece of timber and the number of pieces), Julia did not show any signs that she thought that what she had done was in any way inappropriate, nor did she show any interest in reconsidering her answer. Asked to determine how many days it took Incy Wincy spider to climb a wall if he managed to climb up 5 m per day but slipped back 2 m every night (multiple-step problem, not enough information), without hesitation Julia immediately said: "ten". Asked why she chose to do that, she said: "Cos'..I multiplied with that one and that one is like the question you've asked me just then".

Towards the end of the training sequence Julia was given the mass, height and eye and hair colour of a boy and his sister. Asked: "can you tell me how much older the boy is than the girl?" Julia pondered for a moment and then wrote down all the measurements, checking the accuracy of the last one only. She then subtracted the girls mass from her height, the boys mass from his height and then found the difference between the two results: "they are 25 years apart". Asked if she thought this could be right, Julia replied: "no...but that's all you can do". Incidentally, every other student interviewed on this item rejected it on the basis that the critical information was not there, although some took a long time to make up their minds.

At the end of the ten week period, Julia's attitude to mathematics had not changed very much and her performance on the post test set of problems indicated that her primary concern was still with establishing which of the four operations were required. It was very obvious that, with the exception of a one-step problem in which too much information was provided, Julia had not engaged in any sort of activity to help establish meaning. Numbers seemed to be retrieved more or less at random and then an educated guess made as to which operation was required.

One very encouraging trend reported across all four grades was the change observed in students previously categorised as being in "the slower group". End of year test results (quite independent of the teaching experiment) indicated that these children did far better than on previous class tests and that this seemed to be due to the fact that drawings, estimations and questioning techniques were being employed effectively.

The teacher's diaries indicated significant improvements in attitude and confidence across all ability levels (with a few qualified exceptions as in Julia's case) and noted the greater participation of "quieter" students in discussion sessions. Teachers felt that students were far more likey to check or comment upon problem structure (number of steps and the amount and nature of information provided) and seemed far more willing to accept open-ended problems. One teacher commented that students "became less compulsive and more reflective about the approach they were using".

My observation was that the teachers behaviour also changed quite significantly over the ten week period.

6. Conclusions.

It would seem that metacognition, as it is defined is, a force governing children's mathematical problem solving and that, at least for certain types of students, enhanced metacognition can be brought about by training. However the suggested dichotemy between cognitive awareness and and self-regulation is no where near as simple as it first appears. Case studies such as Julia's suggest that there is a complex interaction between conceptual and procedural knowledge which is heavily influenced by systems of beliefs and values. A much longer study is needed to determine: if these factors can be isolated and described more accurately, the extent and nature of the interaction between cognitive awareness and self-regulation and the extent to which this sort of behaviour can be described by recent theoretical developments in cognitive science.

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Mathematics Teaching
A Specification in a Constructivist Frame of Reference[1]

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In constructivism, mathematics teaching consists primarily of the mathematical interactions between a teacher and children (Steffe, L. P. & von Glasersfeld, E. 1985). In the course of a mathematical interaction, the teacher acts with an intended meaning and the children interpret the teacher's actions using their mathematical schemes, creating actual meanings. The teacher must infer these schemes based on the language and actions of the children and then make decisions about what to include in possible zones of potential development (Vygotski, 1934) of the children. This general orientation to mathematics teaching requires detailed specification in experimental teaching before it can be useful to the practicing teacher of mathematics. The question of what constitutes children's mathematical schemes and possible zones of potential development with respect to those schemes remains to be worked out in research. If practicing teachers become knowledgeable about children's mathematical schemes and possible zones of potential development, they could create problematic situations that fit the children and know what learning the situations might provoke in the children.

In the context of the first year of a two-year teaching experiment, we have been investigating the multiplying and dividing schemes of six eight-year-old children and how the children might modify these schemes. To demonstrate our method, we analyze the multiplying and dividing schemes of Maya and the decisions that we, as teacher and observer, have made concerning her zone of potential development during the course of two teaching episodes held on 23 and 30 April 1986. We first specify her dividing and multiplying schemes that we observed in the 23 April 1986 teaching episode.

Dividing as double counting backward

Maya's scheme for dividing by three was to count backward by one, take each trio of number words as a unit, and count those units of three. This operative scheme was used independently by Maya to solve a task where we placed 21 numeral cards in a row in front of her and then hid them.

> T(Teacher): If you start from there (the beginning of the covered row) and take three cards at a time to make a pile, I wonder how many piles of three could you make? M(Maya): (Sits silently in deep concentration for approximately two minutes) Seven! W(Witness): When you counted, what did you say? M: 21, 20, 19--that would be one; 18, 17, 16--that would be two; etc.

This scheme was not suggested by her regular classroom teacher nor by us. It was what we call a child-generated dividing algorithm (Steffe, 1983; Hatfield, 1976). Maya's division concept can be explained as follows. "Twenty-one" seemed to be a symbol for the *number word sequence* from "one" up to and including "twenty-one" which, in turn, symbolized a *number sequence*[2]. "Three" seemed to be a symbol for a more general unit in the sense that she could implement it using any three number words. This rather arbitrary nature of three served as the basis for her potentially repeatable operation of "take three out of twenty-one"--an anticipatory operation. "Twenty-one divided by three", then, meant to make a unit of three--{21, 20, 19}; then another; etc. In this case, we call three *repeatable*. Seven, the number of times that three was repeated, was a *result* of actually operating.

Lack of division as an inversion of multiplication. At this point, we made a decision to encourage Maya to formulate the results of her dividing scheme in terms of her multiplying scheme.

> T: Can you give me a multiplication problem for that? M: Twenty-one times three? T: What does twenty-one times three mean? M: Twenty-one, and take three out of twenty-one. T: Is that twenty-one times three or twenty-one divided by three? M: Twenty-one divided by three! T: Can you give me a multiplication problem for that? M: (Sits silently for over one minute.)

T: What are you doing? M: I am figuring out how many threes equal seven! Maya's dividing scheme appeared to be unrelated to her multiplying scheme because she did not take the result, seven, as a starting point--as how many times that she could repeat three--and 21 as its result. Maya did not see how the task could be explained using multiplication and seemed to have little awareness of the structure of her operations nor of the possibility of unpacking the seven units of three into their constituent unit items.

Multiplying as double counting forward

The teacher eventually asked Maya what "five times eight" meant in an attempt to understand her multiplying scheme. Maya could not say, so the teacher asked Maya what she would do to find "five times eight". Maya said that she would count by five eight times! However, her multiplying scheme was still not activated in what she took as dividing situations. Maya's multiplying procedure of counting by fives, while connected to expressions like "five times eight", did not seem to be based on the operation of combining several units of equal numerosity into one unit nor of the operation of unpacking those units into their constituent unit items. Her multiplying scheme was the result of her necessity to learn the multiplication table in her regular classroom and was an enactive scheme--it was not an object of reflection.

Lack of combining units of three. We attempted to further understand Maya's concept of multiplication in the 30 April 1986 teaching episode. We wondered if she could combine six units of three and five units of three when she was asked to find how many units of one were in the result. To begin, we asked Maya to count out eighteen blocks by three.

> T: Put eighteen blocks into that container. You can count by three if you want. M: (Maya takes three blocks at a time and places them into the container). T: Give me a multiplication problem for that. M: (Long pause) three times six equals eighteen.

One essential difference in this situation and the preceding situations that

Maya took solely as division seemed to reside in the fact that she actually counted out eighteen blocks by three--the collection was not given as in the previous situations. Rather than segment her number word sequence into units of three number words that she then counted, she formed the collection of blocks by counting by three.

To continue the exploration, the teacher asked Maya to put fifteen blocks into another container. Maya put them in by three and said that she had five groups of three in the container. The teacher then poured the contents of the two containers together and asked Maya to find the number of blocks in that combined collection of blocks using her units of three.

The only answer that Maya could give was "thirty" (she knew that "five times six is thirty"). Her failure to combine the two lots additively is an excellent indicator of the nature of her concept of multiplication. Maya could make composite units of three and then re-present the process and take each composite unit of three as one thing, forming what we call abstract composite units of three. We also believe that she was aware of the abstract composite units of three in the two lots of six and five. However, when she was asked to find the number of blocks in the combined collection, a lacuna appeared in her reasoning. She did not take her abstract units of three as material for further operating, combining them and then somehow transforming them into units of one. These very complex operations were beyond Maya's zone of potential development at the particular time of the teaching episodes.

Comments

Currently, we can only speculate on the nature of Maya's multiplication and division concepts that will allow us, as teachers, to include division as the inversion of multiplication, distributive reasoning, and more sophisticated multiplying and dividing procedures (that are part and parcel of her concepts) in a model of her zone of potential development. One of our hypotheses is that it will be necessary for Maya to develop a more sophisticated unit--an iterable unit--and the corresponding iterative concept that we have observed in Tenryn, another child of our teaching experiment. Behaviorally, iteration looks exactly like repetition. But there is a profound difference because iteration is a manifestation of an underlying iterable unit and a system of flexible operations. A key to Maya's development of iterable units may be her use of number sequences in a context similar to her application of her dividing scheme.

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2. A number sequence is taken as the items of a composite unit of specific numerosity that can be produced by counting (Steffe, & von Glasersfeld, In Press).

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Composite Units and the Operations that Constitute Them

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Euclid defined natural numbers as "multitudes of units" and in many arithmetical operations such "multitudes" are themselves treated as though they were units. In multiplication or in division, for instance, when we say, "Five times three is fifteen" or "How many times does five go into fifteen?", the items referred to by "three" in the first and by "five" in the second question, are treated as units but differentiated by the fact that the one is itself composed of three units and the other of five. The ability to conceptualize a number in these two ways, as a unit and as a composite, and to switch from the one conception to the other, is crucial in a great many mathematical activities. Children are not born with this ability but have to learn it. Though there are behavioral indications that tell an observer more or less reliably when a child is able to switch from the composite concept to the unitary one, we have little if any idea of the actual conceptual operations that are involved. The problem is often dismissed or not even acknowledged. This suggests that the relevant conceptual operations are not exclusive to the realm of numbers, which is certainly interesting but does not get us any closer to useful inferences concerning the operations that have to be carried out.

In earlier studies we have shown that the items children are able to count at the beginning of their arithmetical careers are "concrete" perceptual objects and that it is only through the development of visualization and the child's growing awareness of his/her own activities (motor acts, verbalizations, and finally conceptual operations) that an "abstract" concept of number is attained (Steffe, von Glasersfeld, Richards, & Cobb, 1983; von Glasersfeld, 1982). More recent studies, focusing on children's first attempts at multiplication, suggest that there may be yet another sequence of relatively discrete steps that have to be made before arithmetical operations thatinvolve both the composite unit of a number and its components become easily accessible.

What makes it difficult to *see* a number as both a unit and a composite of units is, in principle, not very different from the difficulty encountered at the intersection of two or more classes. To realize that one and the same bead may be considered a "red bead" as well as a "wooden bead", requires the ability to *switch from one way of seeing to another*. To *know* that "five" taken three times brings one up to fifteen, one must be able to *see* that each of the "fives" is treated as a unit when it is taken three times, but must be treated as a composite of five "ones" so that a count of all the components will yield "fifteen".

Though we may not be aware of it, we constantly depend on this ability to treat one and the same item in two different ways in the area of verbal communication. Language is used to convey "meaning" or, as it is often called, "information". But how much of the potential meaning of a linguistic expression do we actually realize? If one of us, L, says to E: "I have to go to class now, we'll have to discuss this later," E is unlikely to construct for himself a detailed representation of what the word "class" means in this context, or to visualize what L will be doing in class. He *could* do it, but he is not likely to--unless there is some special reason. Under normal circumstances, he will register the message in its verbal form without having interpreted it experientially beyond the simple fact that the conversation will have to be postponed. Yet, if the need arises at some later point, E would be able to reconstruct the scene of L going to class in considerable detail. The point is that a word, or a linguistic expression in general, has the potential of being interpreted in terms of actual representations, but can also be stored *without* the interpretation having been carried out by the user.

One *knows* a word if and only if one is able to represent its meaning to oneself; but knowing a word does not entail that one actually calls up the representation for which it stands, every time the word is used. Words more often serve as place-holders, or "symbols", that have the power to call up representations. The salient feature of this dual function of words is that they can, on the one hand, be used as unitary building blocks in the composition of larger structures (phrases, sentences, texts) and, on the other hand, they can always be *unpacked*, in the sense that the representations for which they stand can be built up and brought to awareness. This dual function, however, is not acquired all in one piece. Especially in the case of number words, the facility to switch from the unitary function of the symbol among other symbols to the "expressive" function that involves the representation of a specific number of component units, may take quite a long time to develop.

As with other conceptual skills, we observe the development of their manifestation, but we have no access at all to the actual mechanism that may be responsible for these manifestions. This is the case with "reflecting" and "abstracting", and it is the case with the use of symbols. Nevertheless, we can conceive of a model that provides at least *one* way of thinking about the unobservable process. The model we want to suggest is similar to the one we originally proposed for the construction of unit and number (von Glasersfeld, 1981; Steffe et al., 1983). Its basic feature is the recursiveness of the operation that creates units, so that the products of this operation, i.e., units or groups of units, can themselves be taken as material to form larger units. The neurophysiological underpinnings we hypothesized for the unitizing operation was derived from a theory of attentional frames or pulses that need not concern us here. The aspect we do want to bring out, however, is this: when units of any kind are taken as arithmetical units, i.e., as "ones", and subjected to a further unitizing operation, so that a larger compound structure, a unit-of-units, is produced, this larger unit will be a "number" only if there is a record of *how many* "ones" were united in it. One of the main difficulties, then, in operating with numbers is the management of these records.

In what follows, we summarize what seem to be different steps in the development of the ability to move one's attention from a component unit to a composite one and, at the same time, to keep track of the numerosities involved on each level. One of the difficulties springs from the fact that the numerosity of a collection can be constituted in different ways depending on the counting units that are chosen (fifteen, for instance, can be seen as *three* fives, *five* threes, or *fifteen* ones). Children first learn to count "ones", then units composed of ones, and finally units that are themselves composed of composites of ones ("units-of-units").

The construction of a "number" comprising units-of-units, while requiring the operations that produce a "number" of component units, has its own developmental history. This very difficult composition is not simply achieved in one or even several experiential encounters as recursion of the operation that yields "number" of component units. Rather, the compositions are abstractions based on self-generated activity in contexts that require an awareness of executing the specific operations. It is not possible here to provide the detail that is necessary to understand when and how these abstractions might occur. We only want to note that the abstractions and accompanying conceptual reorganizations have occurred not only in the course of deliberately arranged teaching experiences but also *between* them (Steffe, 1986; Steffe & Cobb, 1983).

The steps towards the construction of a "number" of units-of-units start with an experience of more than one composite unit. When a child counts out, say, twelve blocks by taking pairs, there can be an incipient awareness of a dual experience while the child says "one,two; three,four; five,six; seven,eight; nine,ten; eleven,twelve," in synchrony with making the pairs. The child's focus of attention, however, is predominantly on the unit items of the pairs. Forming pairs is an organizing activity whose results may be only fortuitously recorded through an arrangeent of pairs of blocks that were moved together during the count.

To make records of the component pairs as they are being formed, involves monitoring the counting activity. We find that monitoring counting activity is made possible by the uniting operation of integration (Steffe et al., 1983). The child takes each composite pair as a unit and reords it by counting it or by tallying it, using a marker such as putting up a finger (Steffe, 1986). In either case, the records that mark the units of two created during the counting activity will be either single number words or single finger movements; these unitary items we call *abstract composite units* to distinguish them from the composite units that served as material of the integration.

When a child can re-present the sensory material of an abstract composite unit as a pattern, or by counting, that ability can be used in counting the blocks of, say, seven rows of blocks with three blocks per row, even when the blocks are hidden. In this context, the child can perform what we have called progressive integrations, i.e., intergations involving the results of prior integration. For example, having counted six blocks of two hidden rows (of three) by one, the child can take that as a unit, count three more blocks, and then unite the counted blocks into a new unit of nine. Progressive integrations make possible units like "three more than ...", an extension of three units beyond a unit of specific numerosity.

If the re-presentation of an abstract composite unit is stable , it can be used in a decision to, say, find out how many times that three can be taken out of fifteen. In this sense, three is repeatable. It would be iterable if the child added the unit items of each new unit to the preceding unit items. The difference between the iterable unit and the unit "so many more ..." resides in the child's awareness of his/her own operating. In the latter case, each unit of the three is a result of operating, whereas, in the former case, the units of three are taken as perceptual givens and are only implemented and not constructed.

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INTERACTION BETWEEN GRAPHICAL AND ALGEBRAIC REPRESENTATIONS IN THE USE OF MICROCOMPUTER SOFTWARE

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The interaction between arithmetical concepts, algebraic symbols and graphical representations play a significant cognitive role in the learning of advanced concepts at the high school level. A better understanding of the interactions between these factors may lead to a better instructional design.

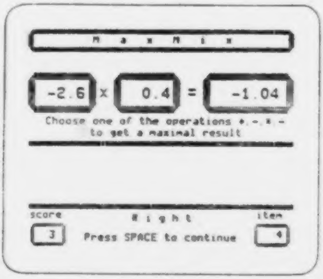
Janvier (1981) investigated difficulties which rise when the concept of variable is presented graphically. Rogalski (1985) studied the cognitive operations which are involved in the identification of points in space by numbers; her research methods consisted of educational microcomputer programs.

The computer offers a variety of representational opportunities for mathematical ideas. It also provides coherent methods for analysing the interactions between the mathematical representation systems.

In this paper, I shall describe a microcomputer package called MaxMix. It was developed to guide students to investigate algebraically and graphically the relation between numbers and operations. The description will be followed by a discussion of four experiments, which were carried on as part of the formative evaluation of the software. These experiments were designed to gain information, on the development of interaction between the various representation systems, by students who used the software.

THE SOFTWARE

The package MaxMix contains four main programs at various cognitive levels and a set of accompanying worksheets. The first program (M1) randomly generates 20 tasks. In each task two numbers are given (e.g., -2.6 and 0.4); the student is asked to choose an operation which will give a maximal result. (In the example one should choose multiplication, see Figure 1.)



Choose one of the operations +, -, x, ÷ to get a maximal result

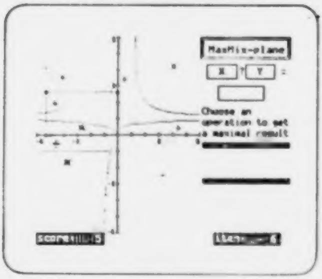
Right

Press SPACE to continue

Figure 1. A MaxMix item

Another version of this program allows the user to control the range from which the numbers are selected and the number of the tasks. This option serves as a tool for the students to perform the investigation assignments in the worksheets. It can also be used by the teacher to organize demonstrations and other classroom activities. We use it as one of our research instruments.

In the second program (M2), the tasks are presented in a graphical format. The two numbers are the coordinates of a point in the plane. It is followed by a program which allows the user to control the parameters and permits further exploration. In the following program (M3), a higher level task in a graphical representation is required, since the numbers are missing. (Figure 2.)



Choose an operation to get a maximal result

Figure 2: MaxMix on the plane

A game for two, Conquer the plane (M4), concludes the package with even higher cognitive tasks and strategies. Only one number is presented and the player has to insert another number and an operation to occupy a certain position. The game requires advanced interaction between the graphical and the algebraic representations. The philosophy which guided the design of the game (see also Taizi and Zehavi, 1985), is that the students will play an active role in the setting of the tasks and consequently it is expected that they will achieve the objectives of the courseware.

THE STUDY

Along with the production of the software an exploratory study was conducted to gather information on the learning processes while using the material. The experiments affected the instructional design of the package.

The four experiments which will be discussed here involved grade 8 classes of the upper ability level students. The programs, mentioned above will be referred to as M1-M4.

The first experiment

Design

	treatment			evaluation
n=26	M1	M1	M1	Q1

The subjects worked three times on program M1 (20 tasks in each application). The third application served for treatment and evaluation purposes. The questionnaire Q1 consisted of 16 questions as the following one:

> Is it possible that the maximal result will be obtained by multiplication if the first number is negative and the second is positive? If YES - give an example.

The most difficult and thus challenging cases were of the two types which are exemplified in Figure 1 and Figure 2. We wanted to see how students encounter these types in three successive applications.

Findings

The average number of wrong responses decreased:

1st application - 11.3; 2nd - 8.5; 3rd - 5.8.

However most of students failed, even in the 3rd application, on items of the difficult types:

Type I -4.8 $\boxed{?}$ $0.6=$ ☐ ; -3.1 $\boxed{?}$ $0.3=$ ☐

Type II -5.9 $\boxed{?}$ $1.9=$ ☐ ; -4.8 $\boxed{?}$ $2.4=$ ☐

In fact 20(!) students failed on all four items. Moreover, 22 students gave wrong responses to the questions in Q1, which are relevant to type I (see the example above) and type II, although they answered correctly in the simple cases.

It was clear that the treatment was not sufficient.

The second experiment

Design

	treatment			evaluation	
GR I n=10	M1	M1	M2	Q1	Q2
GR II n=10	M1	M2	M2	Q1	Q2
GR III n=9	M2	M2	M2	Q1	Q2

The class was divided into three equal-ability groups. We wanted to check the effect of the graphical representation. The naive expectation was that GR III would perform better than the others. The questionnaire Q2 consisted of 18 items as the following examples:

> Insert the missing number so that the given operation will give a maximal result for the two numbers
> (a) .... X 0.5 (b) ....: 2.1 (c) 4.2 + ...

Findings

It was found that GR I did best on M2. It seems that students need some numerical experience before they can gain some advantage of the graphical format. They made mistakes only on the first item of type I and type II and were helped by the graph to give right answers on the items that followed. Students of GR I also did better on the question in Q1 which refers to type I but not to the one on type II. All the groups did not do well on Q2.

The third experiment

Design

	treatment			eval.	treatment			evaluation		
GR I n=16	M1	M1	M1	Q1	M2	M3	M3	Q1	Q2	Q3
GR II n=15	M1	M1	M1	Q2	M2	M2	M2	Q1	Q2	Q3

The fourth experiment

Design

	treatment	evaluation
n=29	M1 M1 M1 - M2 M3 M3 - M4 M4	Q1 Q2 Q3

The questionnaire Q3 and the findings will be discussed in the presentation.

SUMMARY

The findings of this series of experiments illustrates stages of developing interaction between the algebraic and graphical representation:

The numerical treatment of M1 was not sufficient;

The contribution of the graphical program M2 was limited;

The activities of M3 caused just a "rote" use of the graphical picture;

The full treatment M1-M4, at last evoked a meaningful interaction between the representation systems.

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4. MATHEMATICAL CONCEPT FORMATION

Intuitive Mathematics and Schooling in a Lottery Game

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Intuitive knowledge of mathematics, developed outside schools, has been documented among children (Carraher, Carraher, & Schliemann, 1985; Carraher & Schliemann, 1985, Ginsburg, 1977; Groen & Resnick, 1977) as well as among adults (Carraher, in preparation; Lave, 1977; Schliemann, 1984; Scribner, 1984). However, the limits and strengths of this knowledge and its relationship to school instruction are not clear. As suggested by Resnick (in press), understanding these issues may help to deal with the divorcing of formal knowledge from intuitive understanding of mathematics. This study analyzes the relative contributions of practical experience and school experience on the development of mathematical knowledge; the knowledge context is the process of determining bet values in a popular Brazilian lottery game. For this game, five four-digit numbers are draw each day. Bets can be made on the first or on all the five numbers and on its tens, hundreds and/or thousands. A set of bets can be ordered from a "bookie" by stating a number and asking to bet in all the combinations of two, three or four digits on that number. One example of an order by a customer is:

> "I want to bet 2 *cruzeiros* (the Brazilian monetary unit) in the ***inverted thousands and hundreds*** of 583492 (i.e. in all four and three-digit combinations of the digits in 583492), ***from the first to the fifth*** (i.e. in any number to be drawn)."

A bookie who receives this order has to determine the amount of money the customer must pay. To do this, he or she must: (a) find the total number of combinations of six digits in four digit numbers (360) and in three digit numbers (120); (b) add up 360 plus 120, finding 480; (c) multiply 480 by 5 (the number of drawn numbers), finding 2400, and (d) multiply 2400 by 2, finding 4800, the number of *cruzeiros* to be paid. An external analysis of the game suggests that a bookie must be able to solve problems involving the operations described above and to understand the combinatorial system and rules of probability. They often have little or no school experience and learn the rules while betting or while working with experienced bookies. All these aspects make for a most interesting setting for studying intuitive knowledge of mathematics. We analyzed bookies' knowledge of mathematics in the work setting and, later, on problems that differ from those they usually encounter in terms of: (a) the numbers involved, (b) the way questions were asked, and (c) their content.

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Method and Results

Subjects were 20 adult bookies; they were grouped according to school experience. Four of them had never been to school, seven attended school between 1 and 4 years, five between 5 and 8 years, and four subjects attended between 9 and 11 years. Each was observed at work on different days while calculating the orders of ten customers. Later each was asked to solve three series of problems presented randomly, and was then interviewed.

Observations of subjects at work revealed that most of the bets encountered by bookies require fairly simple calculations: 90.0% of the orders specified the amount to be put in each unit and the bookies had to find the number of units and cost of the whole bet, and 60.3% of them involved unit values of 1, 5, 10, 20, or 50 *cruzeiros*. Bookies either memorized the number of possible combinations for each number of digits or referred to reference tables. Memorization accounted for 80.6% of problem solutions. Only two errors occurred in 600 orders. For 57 problems that were not solved via memorization, four calculation procedures were identified; table 1 shows their distribution. One procedure was to simply ask the customer to solve the problem. The second was to use school taught algorithms. The other two procedures were ***iteration*** and ***decomposition*** (see Carraher, Carraher and Schliemann, in press). In the ***decomposition*** procedure the bookie partitions the specified quantities into subtotals that are easier to be calculate and then, after transformations, reunites them. ***Iteration*** is a multiplication problem solution involving successive additions of the same value. ***Iteration*** also allows bookies to avoid division problems: for the few orders in which customers stated a total amount of money to be bet and the bookie had to determine an appropriate unit value, he successively tried out different values through repeated additions until arriving at a result that matched the specified amount. As a whole, no differences that could be attributed to degrees of schooling were found. However, when asked to explain their answers, the schooled subjects provided more elaborate explanations.

The examiner presented problem series *a* and *b* as if she were a customer. Series *a* consisted of six problems stated as they would be by customers but involving amounts that were not round numbers. The six problems in series *b* involved less frequently occurring calculations in which the total amount of money to be bet was specified and the subject was asked to determine the money amounts to apply to each unit. Tables 2 and 3 show that the number of correct answers in each series increases with schooling level. The correlation in both series was significant (Kendall's tau=.30, z=1.85, p=.05 for series *a*, and Kendall's tau=.34, z=2.10, p=.02 for series *b*).

Degree of schooling also affected how subjects approached the problem: while schooled subjects attempted to solve almost all problems and made mostly small computation errors, non-schooled subjects did not even attempt to solve 41.7% of the set problems. Non-schooled subjects consistently solved only two problems in series *b*. An analysis of their solution procedures on these problems together with an examination of their difficulties with other problems suggests that these subjects have learned, through practice, a set of rules corresponding to frequently occurring types of bets that involve round numbers or numbers whose factors are easily found. In both series, all bookies used written procedures frequently. However, these were not used as they would be in school. Instead, writing was a tool to support the invented procedures, such as ***decomposition*** and ***iteration***, when numbers became too cumbersome to calculate mentally.

In series *c*, subjects solved four problems exploring understanding of the combinatorial system. They elicited the same kinds of combinations of entities as found in the betting game but the entities involved were not numbers but colors, letters, or placements in a horse run. Subjects were asked to determine the number of possible combinations, and to state each of them. For 77.5% of the problems subjects admitted, spontaneously or with prompting, that the number of combinations in the problem could be found in the same manner as betting problems are solved, and were then able to determine the correct answer. Among unschooled subjects, however, this percentage dropped to 37.5%. A second analysis classified each subjects answers to three problems that required permutation of three and four elements, in terms of Piagetian stages. These problems were similar but not identical to those devised by Piaget and Inhelder (1951), and they provide a five stage classification system. As shown in table 4, degree of school experience correlates positively to progression through the stages. This correlation was high (Kendall's tau=.58) and significant ($z=3.51$, $p<.001$).

The interviews allowed us to determine whether constant contact with the game provided some understanding of probability rules and of the game's structure. Responses to the interview were coded as logical or empirical/arbitrary. Analysis showed that the mean number of logical answers in each group, increases with school experience (see table 5). Empirical or arbitrary responses reflected the belief that successful bets were a matter of luck. Logical answers connected the probability of occurrence of a specific number to all possible numbers. Since, most certainly, these subjects had not learned probability rules in school, it appears that the contribution of formal schooling is not restricted to topics taught in the classroom.

Discussion

These results suggest that, although mathematical knowledge may develop outside of school training, the use of this knowledge in new situations and an understanding of the relevant mathematical relations embedded in problem-solving rules seem to benefit from school experience. The contribution of everyday practice is evident in the invention of rules and computation strategies as well as memorization of facts--that all provide short-cuts. Further, subjects displayed mathematical knowledge beyond their school experience as a result of everyday practice. However, as Resnick (in press) suggests, this knowledge seems to be restricted to the domain of additive composition. The contribution of school instruction was not found in the use of school algorithms but rather in a more general ability to analyse and understand relationships between the elements in the game. This finding supports Carraher's (in preparation) analysis of how students deal with scale problems, and is similar to results obtained by Scribner and Cole (1981) on the role of schooling versus literacy. Also, our subjects used informal procedures together with modified school taught procedures in a manner similar to that observed by Saxe (1985). This interaction seems to provide the most rapid and least cumbersome way to solve a problem. The development of more efficient strategies in a work environment, already documented by Scribner (1980) and by Schliemann (1984), appears to be a result of a social situation in which many orders must be handled in a short period of time.

The development of implications for school instruction must attend to previous research findings (Schliemann, 1984) suggesting that instruction that was not explicitly related to practice was not used when subjects were asked to solve practical problems. In the present study we have seen that practice without schooling generates knowledge that is limited to the situations in which it originated. The conclusion that naturally follows is that formal instruction on mathematics and problem solving should be presented together with applications to practical, real situations, and these situations should be as varied as possible to avoid restricted knowledge acquisition.

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Table 1

Number and percentage of problems in each group, according to procedure used when the results were not memorized

Groups	School Algorithms	Ask Customer	Decomposition	Iteration
None	0 (0)	4(36.4)	2(18.2)	5(45.4)
1-4 ys	3(17.7)	0 (0)	3(17.7)	11(64.6)
5-8 ys	11(52.4)	0 (0)	2(9.5)	8(38.1)
9-11 ys	3(37.5)	0 (0)	1(12.5)	4(50.0)

Table 2

Number of subjects in each group, according to number of correct answers in problem series *a*

Groups	Number of correct answers						
	0	1	2	3	4	5	6
Non-schooled	1	1		2			
1-4 yrs sch				2	1	3	1
5-8 yrs sch					1	2	2
9-11 yrs sch			1	1		1	1

Table 3

Number of subjects in each group, according to number of correct answers in problem series *b*

Groups	Number of correct answers						
	0	1	2	3	4	5	6
Non-schooled			2	1			1
1-4 yrs sch			1		1	3	2
5-8 yrs sch					2	2	1
9-11 yrs sch					1		3

Table 4

Number of subjects in each group, according to Piagetian Stage when solving problem series *c*

Groups	Stages				
	1A	1B	2A	2B	3A
Unschooled	2	2	-	-	-
1-4 years	4	2	-	1	-
5-8 years	1	1	1	1	1
9-11 years	-	-	-	2	2

Table 5

Percentage of logical and empirical or arbitrary answers in each group

Groups	Emp/Arb	Logical
Unschooled	89	11
1-4 years	84	16
5-8 years	63	37
9-11 years	25	75

The notion of differential for undergraduate Students in Sciences

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I- Introduction

The research presented here started in 1980/81 in the framework of an experimental teaching built with the intention of coordinating mathematics and physics (M. Artigue [1]). The difficulties in coordination which arose at that time and the first empirical analysis which followed led us to some findings, that we summarize as follows :

a) The meaning of the word "differential" is not the same in mathematical and physical teaching :

- In mathematics, differential is primarily perceived as the linear map which approaches best a given map at a given point.
- In physics, differential is percieved as the infinitesimal increment of a physical quantity.

b) The roles played by this concept in the teaching of the two disciplines are actually very different :

- In mathematics, this notion is usually introduced while studying functions of several variables. "Linearity" is emphazised more than "approximation" and the importance given to computation with Jacobian matrices reinforces this fact.
- In physics the differential, rarely explicitely taught, is frequently used about functions of one or several variables. This use takes several forms : approximation, estimation of errors, finding

equations of problems.

These differences in conceptions, classically, make the notion of differential a failure point for any attempt to harmonize the teaching of the two disciplines, at least in France. It is precisely the partial failure of our attempts which led us to develop an interdisciplinary research on that topic, in the framework of the GRECO : "Didactique et Acquisition des connaissances scientifiques" of the CNRS.

II- Presentation of the research

The project has been developped in several directions :

1- Study of the didactical transposition of this concept. (Cf. Y. Chevallard [2]) : How did the current notion of differential take place, historically, in the scientific knowledge ? How did the corresponding teaching objects take shape ? How did they develop ? Where do the differences observed between Mathematics and Physics come from?

2- Study of students' and teachers'conceptions. In particular, how do the students'conceptions integrate the obvious disparity between Mathematics and Physics ? What are its consequences in students'practises.

3- Construction and evaluation of didactical sequences which take into account the results obtained in parts 1 and 2.

The methodology has been adapted to the various aspects considered :

- individual interviews, paper-pencil tests, observation of problem solving by small groups in view of determining students' and teachers'conceptions,

- analysis of curricula, textbooks and papers published in journals such as "l'Enseignement Mathématique", in view of studying the evolution of teaching,

- experimentation of didactical sequences in lectures and in exercise sessions.

The research is currently in process. No questionnaire has been passed on a great scale yet. However, the coherence of the partial results obtained in various populations, already indicates some streamlines. In this talk, we shall focus on point 2 above, specially on students'conceptions.

III- Students'conceptions. Some findings

A- In France, in secondary school (up to 18), for more than a century, the main notion has been that of derivative, classically introduced in 1ere (17 years). It was presented, up to 1970 as a quotient limit. Since 1971, the notion of linear approximation has become dominant. The word "differential" appeared here and there in curricula. Nowadays, it occurs just in "differential notation for derivative".

In October 1984, 122 students in first year at the University were asked the following questions, during exercise sessions in Mathematics :

"Do you know the notations $\frac{df}{dx}$, df, dx ? If yes, where did you meet them ? With which meaning ?

Results :

The notation $\frac{df}{dx}$, known to nearly all students (99 %) is just synonimous to derivative. Only seven of them refer to differentials. The notation df is known to 52 %. Concerning its meaning, the answers are often unclear. Apparently, many students, though feeling the need of making a distinction, have difficulties in finding a meaning for df different from that given to df/dx. Eleven students refer to differentials, two refer to a small increment of f. The notation dx is more familiar (68 %), essentially because it occurs in integration. There, it is perceived as an indicator of the integration variable. However, nine students mention an interpretation as "differential of the variable x" and ten refer to a small quantity. A similar questionnaire was proposed to other students, at the same level, in an exercise session, in Physics. A much stronger tendency to consider df and dx as small increments was observed, especially among those who take the class for the second time.

B- In University, in Mathematics, differential is introduced as a linear map, first about functions of several variables. It should be noticed that such a definition is rather recent in the context of the scientific knowledge (Cf. Stolz [3], Frechet [4]). Even more in teaching. It appeared in curricula about 1960, taking the place of the more classical presentation in terms of first differential and total differential.

35 students in third year at the University attending the class of differential calculus were presented with a questionnaire. Given the question : "Which definition of a differential would you give to a student in first year ?", they answer in large numbers by proposing a definition as linear map. Five of them reduce the notion to that of Jacobian matrix. This reduction, itself, becomes strongly dominant in exercises (show that a map is differentiable and determine its differential at a point).

Technically, the computation and use of partial derivatives is apparently well mastered in simple cases. But, the percentage of success collapses in the computation of second order derivatives of composite functions.

This familiarity with the linear and algorithmic settings, shown by the high percentage of answer to this type of questions, contrasts with :

- the deficiency of students'geometrical abilities : only five of them interpret geometrically, in terms of tangent plane the approximation to first order of a function $R^2 \to R$
- their lack of care in dealing with the remainders in expansions

So, in Mathematics, students'strategies rely on linear algebra, algorithms using partial derivatives and Jacobian matrices, powerful theorems such as : differentiability when there are continuous partial derivatives.

In Physics, they rely on approximation, infintely small quantities, and searching for hints. Teaching in Physics does not explicitely stress the difference between procedures seemingly connected to approximation :

- leading to approximate values and error estimates,
- leading to exact values through passing to a limit (derivation or integration).

Concerning the students'conceptions, this produces an assimilation "physics = approximation" which appears in interviews. So the opposition between the two disciplines is reinforced.

For instance, among 45 students in preparation of contests for "Grandes Ecoles", 21 think that the physician presentation for the computation of the volume of a sphere (based on slicing) cannot be made rigorous.

In this context, students elaborate some behaviour rules, close to a naîve use of non standard analysis, which allow them to work automatically without reference to any meaning.

These rules work in relation with linguistic marks. There are blocked, as shown in the research, by suppressing these marks in their usual form. However, this system is pretty performant due to the fact that problems usually given are stereotyped.

In conclusion, two conceptions of the differential coexist in students, linked with different contexts : Mathematics and Physics. It should be noticed that the teaching of these two disciplines favours this pacifical coexistence : the activities proposed are nearly disconnected and each discipline allows algorithmic procedures, which securise students as they don't need any reference to any meaning.

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QUALITATIVE PROPORTIONAL REASONING:
Description of Tasks and Development of Cognitive Structures

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Qualitative reasoning is a significant variable in problem solving performance. Expert problem solvers are known to reason qualitatively about problem components and relationships among them before attempting to describe these components and relationships in quantitative terms (Chi & Glaser, 1982). It is not that expert problem solvers use qualitative reasoning and novice problem solvers do not, but that novice problem solvers direct qualitative reasoning to the surface feature of a problem, rather than structural features, and fail to anticipate relationships between problem components. For instance, when asked to explain the transformation of $2/5$ to $6/x$ with respect to the equality relationship, the novice problem solver may reason the transformation to be additive, and that $x = 9$. Such an argument is inadequate because this particular additive transformation has a qualitiative effect on the ratio relationship which is not taken into consideration.

The consensus of current research is that an expert's reasoning about a problem leads to a superior problem representation because it contains numerous qualitative considerations about problem components and their interactions. Such a problem representation enables the expert to know when qualitative reasoning is inadequate and quantitative reasoning is necessary.

Research about fractions, ratios, and proportions (e.g., Behr, Wachsmuth, Post & Lesh, 1984; Hart, 1981; Karplus, Karplus & Wollman, 1974; Noelting, 1980; Seigler & Vago, 1978) have described correct, incorrect, and inappropriate strategies and inadequate qualitative reasoning that adolescents use for such problems. It has been shown that adolescents frequently use additive comparisons when multiplicative comparisons are required. The effects of qualitative change in the magnitude of components of relationships such as $a/b = c$, are inadequately anticipated by adolescents and adults. Yet, important cause and effect relationships can be deduced through qualitative reasoning alone.

The focus of earlier studies on proportional reasoning has been on whether children have achieved quantitative proportional reasoning. Observed qualitative strategies have been characterized as inadequate without investigation of how they might interact with or serve as a basis for the development of quantitative strategies. One aspect of our research is to investigate whether children have, or can be taught, qualitative reasoning strategies to answer proportionality questions when appropriate, and to determine circumstances where quantitative reasoning is inadequate.

Work by deKleer and Brown (1984) responding to a need to explain the qualitative reasoning observed by expert problem solvers in scientific domains has resulted in a qualitative calculus based on the concept of confluence equation, or qualitative differential equation. In this calculus an equation of the form $a/b = c$ is associated with the qualitative differential equation $\Delta a - \Delta b = \Delta c$ (not to be confused with a standard differential equation). Interest focuses on values of +, −, and 0 assigned to Δa, Δb, or Δc, according to whether a, b, and c are increasing, decreasing, or unchanged (deKleer & Brown, 1984). The qualitative differential equation provides an algebraic description of the qualitative behavior among the three components of the equation $a/b = c$, as shown in the following table.

The Rational Number Project has developed tasks to investigate children's qualitative proportional reasoning. Success on these tasks requires ability to reason qualitatively about two ratio situations modeled by $a/b = c$ and $x/y = k$. They require reasoning about how qualitative changes in a or b and x or y affect c and k, the intensive values of the respective ratios, and of how these qualitative changes affect the comparison of c and k.

Δa Δb	+	−	0
+	?	−	−
−	+	?	+
0	+	−	0

Tesselation Tasks which we developed were based on the concept that any uniform pattern visually expressing a comparative relationship between two or more sets of congruent juxtaposed geometric figures, such as curves to squares, or squares to rectangles, can be used to tesselate the plane. Proportionality problems arise from questions about iterations, partial iterations, or partitions of the basic tesselating pattern. Comparison of rates in terms of equality or inequality relations are embedded in a comparison of two tesselating patterns. Missing value problems are embedded in situations when an iteration, partial iteration, or partition of a basic tesselation pattern is shown with one set of geometric patterns masked. Incorporation of perceptual distractors into a tesselation task serves as a test of the strength of a child's logic for proportional judgments over perceptually based judgments.

The block task is strictly a non-metric proportionality task which involves two pairs of blocks ((A,B) and (C,D)). Corresponding blocks (A,C) and (B,D)) across each pair were constructed from the same kind of unit-blocks; blocks within a pair differed in the size of unit blocks used, A and C using larger units. The number of unit blocks in A and C differed, but remained constant across tasks. The three instantiations of each of B and D compared in number of unit blocks by one less, the same, or one more to A and C, respectively. Subjects were asked to judge the weight relationship between C and one of the instantiations of D based on one of three weight relationships given between A and an instantiation of B.

The three within pair weight relationships crossed with the three within pair number of unit blocks relationships resulted in 27 possible problem situations. Nine carefully selected problems were presented to subjects. The blocks were constructed so that proportional reasoning was required for problem solution, carefully avoiding the possibility of solution by transitive reasoning.

The triangle similarity tasks assume that similarity is a primitive cognitive structure to proportionality. Each task involved right triangles as the pre- and post-image of a uniform or nonuniform right-triangle preserving positive or negative dilation. Three transformations were represented such that from pre- to post-image, the leg of the right triangle in a counter-clockwise orientation to the right angle was stretched or shrunk by an equal, or less, or a greater multiple than the leg in the clockwise orientation to the right angle. Thus, six types of tasks were possible. For missing-value proportionality problems in this context, children were given values of two sides of the transformation pre-image and one corresponding side of the post-image along with a picture to characterize the transformation as one of the six possible, and asked to give the value for the length of a corresponding side of the post-image. For ratio comparison problems, children were given numerical data for two pairs of corresponding sides of the pre- and post-images and then asked to choose from a set of pictures the one of six possibilities which would result if the pre-image were superimposed upon the post-image.

The concept map task is concerned with the conceptual representation of a topic, in this case, proportional reasoning; not with numerical relationships or strategies The children were given a number of concept names on cards and were asked to arrange them so that similar concepts were near each other. They then drew and described links between the concepts which exposed their understanding of the relationship between the concepts. The resulting concept map gives an impression about whether children see relationships between concepts and whether or not they group them hierarchically or as separated entities. The concepts given were general (proportion, fraction) or specific (numerator, denominator). Some were mathematical (rate, ratio) or from applications (speed, time, distance). So there were ample opportunities to find interconnections. The results from this task will be used to aid in constructing semantic nets of the student's cognitive representations of concepts associated with fraction, ratio, and proportion.

These tasks have been given in one-on-one interviews given at intervals in a 15-18 week teaching experiment. Further analysis should help to characterize the development of qualitative and quantitative proportional reasoning strategies, and the interaction between them.

A written test was developed to study the effects of problem context from the two perspectives of qualitative and quantitative reasoning. Quantitative tasks were of missing value (MV) and numerical comparison (NC) types. Qualitative problems contained no numerical values, they required a decision about the directional change of a rate when its numerator or denominator is(are) changed, or to determine the order relationship of two rates according to the variables under consideration.

Four contexts were selected -- speed, mixing, scaling, and density. For each context, separate tests were developed for each of two settings — more familiar and less familiar -- resulting in eight tests. The contexts and settings are listed:

	Setting (more familiar)	Setting (less familiar)
Speed	Running Laps	Driving Cars
Mixing	Mixing Lemonade	Mixing Paint
Scaling	Drawing Classroom Maps	Reading Road Maps
Density	Standing in Movie Line	Hammering Nails

Each test had eight quantitative problems (4 MV and 4 NC) with integer ratios both within and between rate pairs. Two types of qualitative problems (4 rate change and 4 comparison) were included in each test. A qualitative rate change task requires determination of the direction of change in a single rate: "If Nick drove less miles in more time than he did yesterday, his driving speed would be a) Faster b) Slower c) Exactly the same d) There is not enough information to tell." In a qualitative comparison task two rates are given and an order comparison is required: "Bill drove more miles than Greg. Bill drove for less time than Greg. Who was the faster driver?"

Test results from about 950 grade-7 and -8 children were analyzed in a context X setting X problem type X grade level design. Types of student solution strategies were examined for the quantitative problems.

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RECENT COGNITIVE THEORIES APPLIED TO SEQUENTIAL LENGTH MEASURING KNOWLEDGE IN YOUNG CHILDREN

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SUMMARY. This research was designed to determine sequential length measuring knowledge in children aged 3-7 years. Sequences were predicted in advance logically from measurement theory; from a review of the literature; and from the information processing demand of the tasks (cf. Case and Halford). A sample of 80 children from mixed socio-economic backgrounds was tested on measures of capacity to process information and 15 main measurement tasks. Analysis of the data showed that the empirical sequence of length measuring knowledge was most like that predicted from analysis of the information processing demand of the tasks. It is asserted that mathematics curriculum content could be sequenced on the basis of similar information processing analyses.

INTRODUCTION

Lesh and Landau (1983) maintained that cognitive research has produced generalisations that researchers in mathematics ideas have considered to be too crude. They asserted that mathematics educators are now focussing directly on children's mathematical ideas because of their interest in substantive mathematics content and educational implications.

Part of the disillusionment with cognitive research has been caused by the fact that the results of Piaget's work and similar research have not been easy to apply to mathematics education. In the 1960's and 1970's determined efforts were made to apply Piaget's theories to designing mathematics curricula. The effects still persist in many state curricula in Australia and have caused a certain laissez-faire approach to teaching mathematics to young children in particular.

With the benefit of hindsight we can see that Piaget's theory could not be directly applied to teaching mathematics. He was concerned with the logic of thinking and with providing a formal logical description of human knowledge as it developed over time. Because of his structural descriptions of thought and the tasks he designed, his data present a picture of deficits in the thinking of pre-operational children in particular. Siegler (1981) suggested that mathematics educators should look more closely at the knowledge that young children do possess without feeling obliged to fit it to a preconceived Piagetian framework. The research in mathematics education appears now to be taking up such a challenge.

Recent cognitive research however is not such a "crude" tool for mathematics educators as Lesh and Landau suggested. In particular the work of Halford and Case has much to offer to mathematics education. Each of the theories is concerned with the notion of an increasing upper limit to children's capacity to process information. Case (1985) and Halford (1982) have employed a variety of tests to assess that capacity at approximate ages. Halford (1982) has described sequential levels of children's ability to relate symbols to environmental elements and identified classes of concepts, belonging to each level (Halford, in press). These include mathematics concepts and provide a basis for determining the demand on processing capacity of content and hence sequence in mathematics curricula. Case (1985) has described tasks that children should be able to perform as age and M - space increase.

The research described in this paper was intended to determine children's sequential knowledge of length measuring. It was designed to predict the sequence of development of knowledge of a particular mathematical idea from recent cognitive theories and to compare the predictions with empirical research.

Three alternative sequences of length measuring knowledge were posited before children were tested. The first was a logico/mathematical one based on measurement theory. The component skills (variables -V) required for length measuring were determined and ordered according to whether they were logical prerequisites for subsequent skills. Research findings for each aspect of length measuring, identified in the logico/mathematical analysis, were reviewed. The second sequence of skills was based on that review. The skills were ordered according to the approximate ages at which 50% or more of children apparently demonstrated mastery. The third sequence was based on analysing the demand that each of the skills, identified in the first sequence, would make on children's capacity to process information. The skills were then sequenced a third time according to their hypothesised demand (cf. Case and Halford in particular), the norms for increase in capacity to process information, and classes of tasks possible at each level. The three sequences are shown in Table 1 of the longer paper provided at the session and described there on the basis of a selective discussion of the references.

METHOD

Sample

This consisted of 80 children from kindergartens and primary schools in the southern suburbs of Adelaide, South Australia. There were 8 boys and 8 girls at five age levels (mean ages at each level were 3:7, 4:6, 5:6, 6:5 and 7:5). The sample included equal numbers of children from low, middle and high socio-economic backgrounds.

Procedure

Each child was withdrawn and tested individually in two separate sessions. The tests included an M space measure, two measures of short term memory and tests for variables as listed below. The tasks are described in detail in Boulton-Lewis (1983) and briefly in the paper available at the session.

M-space was measured with the Card Counting Test used by Case (1977). The repeating of digits test from the Stanford-Binet Intelligence Scale (adapted from Terman and Merrill, 1964) was used to measure digit span to determine short term memory. A repeating of words test was devised for this study as an additional measure of short term memory.

The other tests were as follows; V 8 Subitizing, V 11 and V 12, number names in sequence from 5 to 10, and 10 to 100. V 14, One-to-one correspondence to determine equivalence and cardinal number (and also V 9, enumeration, V 10, number names to 5, V 13, a rule for counting.) V 15, Relative magnitude of small numbers (4 > 3, 7 > 6). V 16, Recognition of equality/inequality of length. V 17(a), Ordering of lengths pair by pair and/or V 17 (b) seriation. V 18 Construction of horizontal lines. V 19, Diagonal lines and V 20 recognition of a straight line. V 21, Recognition of length invariance (without explanation) and V 22 conservation of length invariance. V 23, Correct resonse to transitivity task and V 24 transitive reasoning. V 29, Measure with a standard device (without explanation) and V 30 (with explanation). V 34, Counting to 10 with a rule. (Sum of V 9 Enumeration, V 11 and V 13.) V 35, Length measuring strategy using arbitrary units (sum of V 25, V 26, V 27 and V 28.)

RESULTS

The results are summarised in Figure 1 of the paper available at the session. Figure 1 is a model of the sequence of development of length and number knowledge in children aged 3 to 7 years. It was constructed from analyses of the data through the SPSS Guttman and Guttman-Lingoes Multiple Scalogram scaling procedures, cross-tabulation of the variables item by item, and phi and McNemar chi-square statistics.

It can be seen from Figure 1 that the variables form three or four clearly identifiable groups. Variables 18, 16, 17, 9, 8 and 10 are the first concepts to develop. There was no significant chi-square value between any of these variables which would indicate strong directionality although the scalogram analyses give the sequence of development for these skills.

The second group of variables includes 21, 34, 15, 22, 23 and 19. There was a clear one way association between the last variables in the first group (17 and 8) based on the chi-square value, and the first variables in the second group (21 and 34).

The third group of variables includes 29 and 14. Variables 30, 35 and 24 could be considered to be in this group or in a further fourth group. Variable 24 certainly is learned much later than the other variables. Variable 29 occurs much later than predicted from the literature review or from the analysis of demand of the tasks.

The sequence of the variables in Figure 1 was most like the sequence predicted from the analysis of the information processing demand of the tasks. The only significant exceptions were variables 29 and 24, both of which are apparently learned later than predicted.

The variables were also cross-tabulated with sex, measures of M-space, short term memory span and age. The 50% success rates for age, M-space and span showed similar groupings to those in Figure 1. Cross tabulations by sex showed no significant difference in performance on any of the tasks between boys and girls. Other statistical analyses are discussed in the fuller paper.

DISCUSSION

The data from this research showed quite clearly that the sequence for acquisition of length knowledge is closely related to developing capacity to process information. Increasingly complex knowledge of length measuring was predicted and shown to develop with increasing M space and short term memory. The children younger than 6 years in the sample had no direct instruction in measurement concepts. The older children may have benefitted from school instruction to the limit of their capacity.

It can be seen from Figure 1 that there are levels of knowledge for concepts such as invariance of length and cardinal number. They do not develop in an all or none fashion. Rather levels of knowledge of those concepts appear to be closely related to capacity to process information.

Because the sequence shown in Figure 1 is most like the sequence predicted from the analysis of the information processing demand of each task it is asserted that the theories of Case and Halford are potentially very useful for curriculum planning. A suggested procedure for choice and sequence of curriculum content is outlined in the further paper available at the session.

REFERENCES

All other references are listed in the full paper available at the session.

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RATED ADDITION: A CORRECT ADDITIVE SOLUTION FOR PROPORTIONS PROBLEMS

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Several authors have documented the existence of mathematical abilities which rely upon problem-solving procedures that differ from those taught in schools. These abilities have all been in the realm of arithmetic operations. Child-invented methods observed for dealing with more complex problems like proportions have been of limited application and less successful (e.g., Hart, 1981) than those reported for aritnmetic operations.

The present study describes a successful intuitive method for solving scale problems. The participants in the study, construction foremen and students, had different previous experience with scales. Foremen deal with blue-prints in their daily lives. They often have to figure out the length of walls using information obtained by measuring the target distance on the blueprint and converting it into real-life size according to the scale used. Their experience with scales is in practical situations, where this numerical transformation is needed and they have previous knowledge of which scales are commonly used. Foremen learn about their professions on the job; no formal training is required. Their learning of scales is accomplished on the job--a situation which strongly favors the development of intuitive strategies for solving scale problems. Students, in contrast, if they learn about scales, do so in the classroom. In Brazil, students usually receive instruction on algorithms before being asked to solve problems by applying the algorithm. Teaching aims at the development of formal mathematical strategies; the rule-of-three is the algorithm taught for the solution of proportions problems.

METHOD

Subjects. Subjects were 17 foremen and 16 7th graders from Recife, Brazil. Of the foremen, three had been in school long enough to have learned about proportions; all had at least five years of practice in the trade. All students had received instruction on the rule-of-three in 7th grade.

Procedure. Subjects were shown four blueprints drawn according to different scales. Two scales (1/100 and 1/50) were familiar to foremen from their practical experience; the others (1/40 and 1/33.3) were unfamiliar. Students had not received any specific instruction on scales; the distinction between familiar and unfamiliar scales does not apply to them.

Subjects, tested individually according to the clinical method, were shown that, for most of the walls, the life size measures were indicated on the blueprint. However, for some of the walls, there were no measures. Their task was to figure out what those measures were by using information from the blueprint. As a start, they obtained one measure from the blueprint and compared it to the life-size measure of the same wall; these two values will be termed the first pair of numbers. The first pair was needed for the application of the rule-of-three and for the identification of the scale by the foremen. The second pair of values was the pair for which there is an unknown, the real-life value, and one known measure, obtained from the blueprint. Table 1 presents a summary of the numerical problems.

Table 1

Problems presented in the Scales Study

Scale	First pair	Known-values in the second pair
1/100	3 cm/3 m	4 cm; 2,8 cm; 3,2 cm
1/50	6 cm/3 m	9 cm; 7,5 cm; 5,5 cm
1/40	5 cm/2 m	8 cm
1/33.3	9 cm/3 m	15 cm

RESULTS

Three types of results were analysed: accuracy of solutions, strategies in problem solving and types of errors.

With respect to **accuracy**, foremen did better than students with both familiar scales and the 1/40 scale; students did better with the 1/33.3 scale.

With respect to **strategies used in problem solving**, results showed that: (1) only one student used the rule-of-three; (2) additive solutions (see Karplus et al, 1983; Hart, 1981; Inhelder and Piaget, 1951), in which the difference between the first pair is kept constant in the second pair, were observed in only two of the 259 observations (8 problems x 33 subjects; 5 missing observations); (3) the most common method of solution was one which can be termed "rated addition", according to which the subject finds a simpler ratio (usually 1/x) and then proceeds by adding corresponding amounts to each member of the pair of numbers. This method must be distinguished from the additive solutions because, contrary to those, it maintains the ratio constant.

The methods used in finding the simpler ratio were of two types: (1) hypothesis testing; and (2) identifying the relation. **Hypothesis testing** was a strategy used by foremen which treated the known scales as a pool of hypotheses and proceeded by eliminating/accepting a scale by comparing calculations for the first pair with the given values. If the calculations according to the hypothesized scale checked with the given values, the hypothesis was accepted; otherwise it was rejected. This stretagy can only work for familiar scales because unfamiliar scales are not part of the pool of hypotheses. Hypothesis testing was not a strategy available to students because they did not have a pool of familiar scales to draw from. **Identifying a relation** was a strategy in which the ratio 1/xwas obtained through multiplication/division within the first pair. This strategy, which works for any scale, made foremen "uncomfortable" with the scale 1/33.3 because of the definition of scale used in their trade; nonetheless, none of the foremen who were able to identify this ratio failed to solve the problem. Students worked with this scale as 3/1 as they were not subjected to the same type of expectations on how to name scales. The distribution of responses according to strategies is presented in Table 2.

Table 2

Percentage of subjects by type of strategy

		Scales	
	1/50	**1/40**	**1/33.3**
Foremen			
Hypotheses testing	47	25	35
Identifying ratio	47	67	59
Other	6	8	6
Students			
Identifying ratio	88	82	82
Other	12	18	18

Among foremen, there was no relationship between level of schooling and type of strategy used in identifying the scale; two illiterate foremen solved all the problems, finding the simpler ratio by division/multiplication, while four with more than five years of schooling did not.

Error analysis showed that students difficulties resulted mostly from the interpretation of results; after correct calculations, they read the results incorrectly. They also made mistakes in working with decimals, coming up with responses such as "three point seven point five" (calculating the life size for 7,5 cm in the 1/50 scale) or "three point seven and a half". Foremen, in contrast, had almost perfect performance on the familiar scales and displayed no errors of interpretation but, like students, had difficulty in dealing with decimals.

CONCLUSIONS

An intuitive strategy for dealing with scale problems was documented, replicating previous observations (Abramowitz, 1976, and Rupley, 1981; in Karplus et al, 1983). This strategy involves two steps, (1) finding a simpler ratio, and (2) applying it additively to compute the result. It remains close to children's understanding of the problem and does not introduce computational rules which may seem arbitrary to children, like those in the rule-of-three. It is suggested that children could perhaps learn more about ratio and proportions if encouraged to solve problems through rated addition and reflect upon the meaning of this way of adding.

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Concerning conceptions of area (pupils aged 9 to 11)

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By the dialectic relation they establish between space and numbers, spatial measures (length, area, volume) play a key role in the conceptualization of these two fields. Among spatial measures, the measure of area play a priviledged role in the building of multiplicative structures. This from two points of view : numbers operate on areas and areas appear as products of lengths.

One knows the difficulties for some pupils to conceive the measure of the area of a triangle in square centimeters.
These difficulties can be related to the conception of areas associated to the existence of an actual tesselation of the surface, and to the shape of the tiles.

The problem is then to develop learning situations which allow confrontation and evolution of conceptions, so as to master these difficulties.

We have built didactical sequences which satisfy hypotheses stated later, we have observed their development in two classes : a CM1 (27 pupils 9-10 years old) and a CM2 (22 pupils 10-11 years old), and we have detected the conceptions implemented by these pupils in two occasions :

1) during individual interviews, at the end of the Second Trimester for the CM1, at the end of the school year for the CM2.

2) during paper-pencil tests, at the end of the school year, for the two classes. The tests were those used by J. Rogalski in CM1, CM2 and 6^e(11-12 years) [Cf.Recherches en Didactique des Mathématiques,Vol3.3].

I- Our didactical choice for constructing the learning sequences

• Theoretical framework

We resume the hypotheses stated in [R. Douady 1984, thèse and PME 9] :

1) On the strictly cognitive ground, the implementation of unbalances and the possibility of reequilibration by pupils can be obtained, for an important part of the concepts, within appropriate "interplays between settings" by using problems to be built ad hoc.

The reequilibrations we are aiming at correspond to the acquisition of new concepts involved in "Tool/object dialectics".

On the didactical ground, we make the hypotheses that there are some adequate problems which allow the realization of the above hypotheses, and which can be part of a global organization of the teaching efficient for most pupils.

Here, for the measure of areas to play fully its role in the construction of their knowledge, pupils have to make three settings interact : the setting of surfaces (geometry without measure, but with movements and transformations), that of magnitudes (lengths and areas), and that of numbers. This involves the necessity of distinguishing these three settings, and also the possibility of identifying two of them - or the three - if needed, according to the problem.

• Presentation of the notion of area

The area is a way of accounting for the room occupied by a surface in the plane. The equality is defined from 3 points of view :

- . The area is invariant by moving.
- . It is also invariant by cuting in pieces and patching them back without loss or overlap.
- . On a grid, two surfaces which include the same number of squares have the same area.

In the same time, comparison of areas is defined using inclusion of surfaces, and addition of areas using juxtaposition of surfaces.

The three points of view above make it possible to compare areas in some cases but not all. For instance you cannot find a square with the same area as a given disc. A way of answering this is to use the measure. Another way is to call on another magnitude which can be measured : mass.

A description of the didactical sequences we worked with can be found in [Douady-Perrin 84-85, Petit x N°6-N°8 IREM Grenoble].

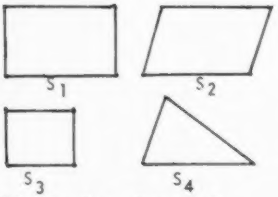
The main points are : comparison of areas without measuring, differenciation area/length, approach of measure in two steps : tesselation, measure of the area of a given surface with different units.

Note that the pupils concerned in this paper only have dealt with a part of the sequences described there.

II- Conceptions of the pupils

1) Interviews

We have interviewed the pupils by groups of 2 in order to observe, and sometime induce, conflicts of conceptions. They had to compare the areas below



The pictures were drawn on white paper. Under request, squared paper was provided.

Strategies observed

a) reduce comparison of areas to comparison of numbers

- using a grid and counting squares (on squared paper or squaring the white paper, with squares or with rectangles).
- Multiplying lengths to compute areas, inventing wrong formulas for triangles or parallelograms.

b) reduce comparison of areas to that of sides (S_1 and S_2)

c) cut and patch in an appropriate way (to compare S_1 and S_2, or S_3 and S_4) ; or pave (S_1 with S_3, S_2 with S_4).

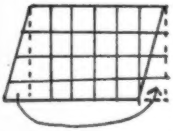
d) "Straigthen" the parallelogram into a rectangle ; lean the parallelogram even more.

Conflicts and changes of strategy

Strategy c on one hand, strategies b and d on the other, conflict since they lead to opposite conclusions. This surprises the pupils, since they expect to get the same results.

They measure the height of the parallelogram, and are amazed to find it shorten than the side.

For some of them, this is a way out of the conflict. These use strategy c. Others try to pave the rectangle and the parallelogram.



with the same squares. They are led to put together the two beaks of the parallelogram, and find 24 squares for both surfaces.

For other ones, the change of stràtegy takes place when they are asked to "lean the parallelogram more and more", until it becomes clear that the area has turned very small.

Our interpretation is that these pupils amalgamate three transformations :

- the "jointed parallelogram", which preserves the length of the sides, but not the area.
- the sliding of a side (on the line which carries it), which preserves areas but not lengths of sides
- The rotation around one vertex, which preserves all lengths and areas.

Remark : In the class CM2, where more time had been devoted to the study of area without measuring and to its invariance under cut and patch, strategy c has been chosen most, including to compare S_3 and S_4. Difficulties in handling have induced changes of strategy in this case : either to calling on measures (sometimes wrong), or to refering to S_1 and S_2

2) Paper-pencil tests

We have proposed questionnaires used by J. Rogalski [Recherches en Didactique des Mathématiques, vol 3.3], including "S.N. without an example of tesselation" :

- How many small squares (resp. rectangles, triangles ...) in a big one ?

- How many paint cans are needed to paint the big square (resp.), knowing what is needed for a small one ? (the ratios are 2 or 3).

J. Rogalski gets answers to these questions which are different for the square (mastered in CM1) and for the triangle : "a long evolution is needed before the representations of tesselation of the triangle are at disposal (end of 5^e : 13 years). Only at the end of 4^e (14 years) these representations are available for 3/4 of the pupils". For spontaneous tesselation (question 1), she gets 48% (resp. 39%) correct answers for the equilateral triangle with ratio 2 (resp. 3), all classes together (9 to 14 years). For these questions, we get answers comparable to those of pupils in 4^e (13-14 years). Note also that correct results are in the same order of magnitude, whether squares, triangles or parallelograms are concerned, whether the ratio is 2 or 3. The rate of passing from a correct answer in tesselation to a correct answer for paint (success (paint + tess) / success tess) ranges from 85% to 100% in CM2, from 58% to 90% in CM1, and is better for triangles than for other surfaces. These rates are comparable to those of J. Rogalski 2 years later.

In conclusion, the work done in didactical sequences has apparently made available the tesselation of various surfaces, so that the pupils become able to answer some classical questions concerning areas.

These didactical sequences have been completed later by other one in order to implement the dynamical point of view induced by geometrical transformations.

Children's Perception of Fractions

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The research referred to in this paper is part of the project Strategies and Errors in Secondary Mathematics, based at Chelsea College, London University from 1979-1983.

In the case of fractions, the main difficulty seemed to be a dependance on half-remembered algorithms, and a failure to appreciate what fractions really are. Thus the decision was made to investigate further children's perception of fractions and, in particular, the models they used when dealing with fractions. The children who took part in the research were aged 12 to 14, and were in middle-ability classes in a number of different schools.

Two aspects which emerged from the research are now discussed briefly.

1. The first relates to children's difficulty with the idea that the fraction a/b can mean $a \div b$.

   When invited to comment on a number of models of the fraction 3/4, the only one with which all children were comfortable was the part-whole one. It became apparent that the commonest reason for rejecting other models was because it was seen not to fit in with this view. For example, a set of 4 counters, of which 3 were coloured was rejected because "It wouldn't be right, it's not a whole thing" or "No, it's got no shape."

   Of particular interest was the children's response to $3 \div 4$. In most cases the idea that it could have any connection with 3/4 was firmly rejected. What is more, it appeared that these children did not think that an answer even existed when a smaller number is divided by a larger one. There were two strategies frequently employed. Some who read the division the correct way said, for example: "3 shared by 4. You can't do that, 4 is bigger than 3."

The other strategy was to reverse the order of the division, but again fractions were avoided: "3's into 4. 3's go into 4 one, and one remainder."

These children were then asked to look at the pair of divisions 12 ÷ 4 and 4 ÷ 12. Some evidently thought that it was acceptable to treat division as commutative:

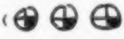
MT "4 12: that's the wrong way round .. 12 won't go into 4"

I. "So what do you think we should do about it?"

MT "Change it round .. 12 divided by 4."

I. "Is it all right to choose which way to do it?"

MT "Yes, because you'd get that one otherwise - 12 into 4 - and that's the wrong way."

When presented with the practical task of sharing 3 cakes between 4 people, almost all the children were able to do this satisfactorily, and in most cases, were then able to see its connection with the fraction 3/4, with the resulting shapes now looking like part-of-a-whole models for 3 quarters () or 3/4 of a cake. One child, though, said: "Ah, I see what you mean. It would be the same that way. But you said these were cakes. But if it was 3 ÷ 4 you couldn't do it."

Part of the teaching programme was designed to give the children experience of the interpretation of the fraction a/b as a ÷ b. This made use of a range of practical tasks, and also of the calculator. One test item asked for responses to both 12 ÷ 4 and 4 ÷12. The main interest in the results lies in the fact that while the number of correct responses increases significantly after the teaching, there remained about one third of the children who thought that they should both have the same answer. Forty-five of the 59 children gave responses which avoided fractions at the pre-test, and 26 still at the immediate post-test.

In a later test-item, consisting of the pair 3 ÷ 4 & 4 ÷ 3, it was not possible to avoid the use of fractions by reversing the order of the division. Ten children thought, at the pre-test, one of the results wwas zero, while 12 thought both answers were the same. By the immediate post-test, these numbers were reduced to 3 amd 10 respectively.

It seems that the perception of a fraction as part of a whole shape, usually a circle or square, is so strongly held by some children that they find it impossible to adapt this model even to include the notion of 3 circles to be shared into 4 equal parts. This restricted view of fractions emerged at several points during the interviews, and where this model of a fraction did not seem to help, the children appeared to abandon it altogether in favour of trying to remember learnt rules.

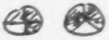
Instances of the limitations of the part-whole model were found when comparing fractions and when adding fractions:

in comparing 5/20 and 1/4, JE chose to draw a rectangle split into twentieths to illustrate 5/20, and compared this with a previous drawing of 1/4 which was a circle divided ito quarters. She said that 5/20 is bigger than 1/4 "because you'd draw a diagram into 20, shade in 5 .. that would be more than 1/4 ... I think 1/4 might be bigger. Hard to tell." And, indeed, it is virtually impossible to compare two fractions of different shapes. This illustrates well one of the problems of the part-whole model, in which, for any meaningful comparison, the 'wholes' have to congruent. In the next extract, JC was working at 2/3 + 3/4, and gave the answer 5/7.

I. "Do you think that answer's all right?"

JC "Yes. If it was the other way round it would be top-heavy"

I. "We have just drawn a picture for 3/4. Could you draw one for 2/3?"

JC "(Draws )"

I. "So we have 3/4 and then we add 2/3 on"

JC "Yes. There are seven pieces and there are five shaded in, two and three."

Not only is the diagram unhelpful, but JC uses it to confirm her error. What is more, there are fundamental objections to using the part-whole model when dealing with any operations on fractions . The notion of adding part of one shape to another part of a second shape is parallel to the attempt to add 2 apples to 3 bananas; to make any sense,it is necessary to think of fractions as numbers, not parts of shapes.

2. One section of the teaching module focussed specifically on the idea that fractions are numbers. The teaching made extensive use if the number-line, and of calculators. One of the test items asked the children to find 3 numbers between 1 and 2 on a number-line.At the first test, ten children said there were no such numbers, whereas only three made this response at the immediate post-test. However, in the second part, the more specific question was asked :"How many numbers do you think there are between 1 and 2", the results were less encouraging. 'Correct' answers included those such as "as many as you like", or "you could go on for ever" as well as "infinitely many". The results are shown below:

Existence of numbers between 1 and 2

Answer	Pre	Post	Del
Correct	6	9	10
None	8	5	6
3	12	13	11
6	4	1	2
1	4	0	1
No response	18	10	9

So, in addition to those who thought that were no numbers between 1 and 2, there was an unusually high group of children who made no attempt, who, perhaps, also found it difficult to accept the existence of any fractions. There were many other occasions when the children failed to accept fractions as numbers : for example, when asked to find a number such that $2 \times \square = 1$, 46% children chose to reply 'there is no number' rather than to attempt an answer or make no response, while 47%

> made a similar reaction to $2 \times \square = 7$.
> This seems to be further evidence thst it is difficult for many children to make the major re-adjustment of thought that is required to accept that fractions are not restricted to geometric illustrations, but are numbers and, as such, an extension to the set of natural numbers with which they are already familiar.

It is the very ubiquity of this part-whole model which is of concern. If the idea of a fraction is, to a child, synonymous with that of a shaded part of a circle or square, then it is difficult to see how they are in a position to make any meaning of adding or multiplying two fractions. There is, even in the case of equivalence or of the ordering of fractions, the underlyimg, but not usually stated, assumption is that these are fractions of the same shape. It may well be that this part-whole model inhibits the development of any other interpretation of a fraction, and that more thought needs to be given to the process by which a child moves to the concept of a fraction as a number.

Proportional Reasoning - Some related situations

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Introduction

The concepts of ratio and proportions, have been the object of many research studies in the last 25 years, because of their importance "in every day situations, in the sciences and in educational systems" and because "the concept of proportion is difficult. It is acquired late. Moreover, many adults do not exhibit mastery of the concept" (Tourniaire and Pulos, 1985).

Touriaire and Pulos who reviewed the research on proportional reasoning, note that there are two kinds of problems:

1) Missing value problems - where three numbers a, b and c are presented, and the task is to find the fourth, x, such that $\frac{a}{b} = \frac{c}{x}$.

2) Comparison problems - where four numbers a, b, c, and d are presented, and the task is to determine whether there is a proportional relation between them.

In this paper we present three situations, related to these two problems, which can be seen as a further contribution to the research into proportional reasoning.

1. Proportional reasoning - as a stereotype

One of the reasons for students failure to solve mathematical problems is their application of an incorrect algorithm. By presenting problems which, if solved by routine algorithms, give rise to unreasonable answers we can create a conflict situation, which may cause some students to question their unthinking use of the algorithm (Markovits et al, 1984).

Example - The "population problem"

> *The area of Belgium is about 30,000 km^2, and its population is about 10 million. The area of Argentina is about 3,000,000 km^2. What do you think is the population of Argentina?*
> *Explain your answer.*

The "population problem" can be (wrongly) considered as a missing value proportional problem, because three numbers are given and the stedent is asked about the fourth.

The problem was given to an experimental group (sixth and seventh graders), a comparison group (seventh graders), to a group of preservice teachers and a group of inservice teachers. The experimental group had worked a unit on estimation and reasonableness of results dealing, among other things, with the fact that not every problem can be solved in an algorithmic way by manipulation of the given numbers. (The treatment did not include any problem where context was similar to the "population problem"). The comparison group and the groups of teachers did not receive any treatment.

The numerical answers are not very important because they tell only a little about what was in students' and teachers' mind. Therefore in the following table, we summarize only the explenations given by the different groups.

TABLE I Explanations given in the "population problem"

Explanation	Experimental group (n=168)		Comparison group (n=76)		Preservice teachers (n=48)		Inservice teachers (n=37)	
No explanation		11%		8%		-		3%
Algorithmic explanation								
- Some manipulation with numbers		5%		16%		-		-
- proportional algorithm		34%		45%		31%		32%
Conflict								
Use of the proportion algorithm but with the concept that the answer is not resonable.	50%	5%	29%	-	69%	-	65%	11%
Correct explanation								
- One cannot know		38%		12%		69%		51%
- Self-knowledge (books, newspapers etc.)		7%		19%		-		3%

A reasonable answer was given by only 29% of the comparison group and 50% of the experimental group. Among the teachers about 2/3 gave correct explanations.

Many unreasonable answers were given. The most popular unreasonable answer was caused by using proportional reasoning in a stereotyped way - about one third of the students in the experimental group and of the preservice and inservice teachers, and almost half of the comparison group.

Examples:

Judith, seventh grade, comparison group:

> *"1000 (millions). Argentina's area is bigger by a 100 time. So we multiply the population of Belgium which is 10, by 100"*

Inservice teacher:

"1,000
10 - 30,000 .
? x - 3,000,000 "

For the teacher groups, proportional reasoning was the only kind of wrong explanations, and it was even expressed by using the "rule of three" whereas the students also had some other stereotypes.

The experimental group responses are better than those of the comparison, so the treatment had some effect, but was insufficient to change student algorithmic attitudes to problem solving. The continuation of such treatment is clearly needed.

2. Absolute and relative error

The concepts of absolute and relative errors are very important in estimation. One of the main difficulties in dealing with this, is to distinguish between the two kinds and to decide which is more relevant in a given situation.

Application of these two errors to comparison problems yielded problems such as the following:

The "errors problem"

> *Noa was asked to estimate the length of the two following segments:*
>
>
>
>
> *(The length of segment a is 10 cm and the length of segment b is 4 cm).*
>
> *Noa's estimate is in error by 3 cm for segment a, and by 2 cm for segment b.*
>
> *Indicate the answer you consider to be relevant:*
>
> *i) Noa's estimate for segment a was better than that for b.*
> *ii) Noa's estimate for segment a was as good as that for b.*
> *iii) Noa's estimate for segment b was better than that for a.*
>
> *Explain your answer.*

The problem was given to the same four groups mentioned in the previous section. For the experimental group, who had a similar problem in the pretest, the "errors problem" served as a posttest.

In order to give a correct answer the student has to understand that the error needs to be regarded in (proportional) relation to the given length, then express the two pairs of numbers (2,4), (3,10) as ratios, and finally to compare these two ratios. The results are presented in Table II.

Table II. Results for the "errors problem"

Answer	Experimental group (n=179)**	Comparison group (n=80)	Preservice teachers (n=47)	Inservice teachers (n=39)
i) Noa's estimate for segment a was better than for b	75% (99%)*	37% (76%)*	49% (100%)*	77% (92%)*
ii) Noa's estimate for segment a was as good as that for b	4%	19%	25%	18%
iii) Noa's estimate for segment b was better than that for a	21%	44%	26%	5%

* The numbers in brackets indicate the percentage of correct explanations in each group.

** In a similar problem given to the experimental group as part of a pretest, only 28% chose the correct answer and only 57% of these gave a correct explanation.

Those who chose iii) as their answer regarded the errors as absolute. For example:

David, seventh grade, comparison group:

> *"I chose the third answer because Noa's error in estimating b was less than that for a. For segment a Noa's error was 3 cm, and for b only 2 cm"*

Most of those who chose ii) misunderstood the problem, and paid attention only to the fact that the estimates and errors were given for both segments.

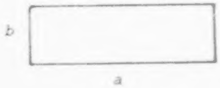
It can be seen from the results that such problems were difficult for the comparison group and for the preservice teachers. The results obtained by the experimental group is encouraging. After a short treatment (including only a couple of problems of this type), which were given a few weeks before the test, the percentage of correct answers and explanations jumped from 16% to 74%.

3. Absolute and relative change

In another study ninth and eleventh graders were asked about changes in perimeter and area as a function of the change in the sides of squares and rectangles. The two following problems consider the idea of absolute versus relative change.

"Rectangle - absolute change problem"

Given a rectangle with sides a and b $(a > b)$



we add x cm to a and remove x cm from b

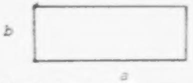
Indicate the correct answer:

i) The perimeter increases
ii) The perimeter decreases
iii) The perimeter does not change
iv) The change in the perimeter depends upon the numerical values of a, b and x.

Explain your answer.

"Rectangle - relative change problem"

Given a rectangle with sides a and b $(a > b)$



we add x% to a and remove x% from b.

Indicate the correct answer:

i) The perimeter increases
ii) The perimeter decreases
iii) The perimeter does not change
iv) The change in the perimeter depends upon the numerical values of a, b and x.

Explain your answer.

In the first problem the change is absolute; so by adding 2x and by subtracting 2x, the perimeter is unchanged. This was the common response among both ninth and eleventh graders.

In the second problem, the change is proportional to the length of the sides, so we add to the perimeter more than we subtract. But many students gave the same answer as in the "absolute problem".

Table III - Responses to the rectangle problem - ninth grade students (n=77)

Rectangle - relative change problem

Rectangle - absolute change problem	Answers	i) Perimeter increases	ii) Perimeter decreasess	iii) Perimeter does not change	iv) The change depends upon a, b, x		
	i) Perimeter increases	-	-	-	-	-	
	ii) Perimeter decreases	1%	3%	-	-	4%	
	iii) Perimeter does not change	8%	8%	57%	17%	90%	correct
	iv) The change depends upon a,b,x	-	-	1%	5%	6%	
		9%	11%	58%	22%	TOTAL	
		correct					

Table III - Responses to the rectangle problem - eleventh grade students (n=58)

Rectangle - relative change problem

Rectangle - absolute change problem

Answers	i)) Perimeter increases	ii) Perimeter decreases	iii) Perimeter does not change	iv) The change depends upon a,b,x		
i) Parameter increases	-	-	1%	-	1%	
ii) Perimeter decreases	-	1%	-	-	1%	
iii) Perimeter does not change	27%	1%	56%	13%	97%	correct
iv) The change depends upon a, b, x	-	1%	-	-	1%	
	27%	3%	57%	13%	TOTAL	
	correct					

More than half the students in both grades treated the "relative problem" similarly to the "absolute" (line no. 3 and column no. 3). For these students percentage change is the same as change in centimetres. Consequently most of the explanations they gave are similar to the explanations they gave in the "absolute" case.

The percentage of students who answered both questions correctly is very low, 27% in grade eleven and 8% in grade nine.

Summary

The problems presented in this paper are a little different from the traditional tasks used in research on proporsional reasoning.

As a continuation of this study we intend to combine the tasks presented here with the traditional tasks and give them to same groups of students and teachers. The comparisons should provide some further contribution to the research into proportional reasoning.

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THE CONSTRUCTION OF THE CONCEPT OF VARIABLE IN A LOGO ENVIRONMENT: A CASE STUDY

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1. Introduction

Several educators share the common feeling that the structure of the Logo language provides a concrete model not only for heuristic notions but also for specific and powerful mathematical ideas. This pedagogical line of thought needs to be investigated so that it can be more precisely articulated and eventually validated.

Many educators have emphasized that, in order to understand a concept, students need to take an active role. Papert (1972) claims that, if children are given a suitable environment in the appropriate, dynamic technology, then they can do real mathematics (rather a just learn about mathematics), that is, they can become "mathematicians."

The concept of variable is of major importance in the development of mathematical thought. In the Portuguese mathematics curricula, as in other countries, this concept is used in an implicit form since primary school. However, mathematics curricula often appear to consider it a "primitive" term universally understood and instantly apprehended by mathematics learners.

Several studies have indicated remarkable weaknesses in children's conceptions of variable (Wagner, 1983 ; Rosnick, 1982 ; Kuchemann, 1981) and suggest that student's conceptual difficulties dealing with functions and equations are related to misconceptions about the variables (Ponte,1984 ; Matos,1985).

I can hypothesise that when children develop Logo activities in an appropriate environment, they are able to construct their own mathematical and programming concepts. My aim was to identify how is constructed a concept of variable in a Logo environment.

2 The subject, environment and procedure

Miguel is 12 years old boy, enrolled in the 6th grade. He works almost daily with his home computer since Spring 1983. At school he's a successful student, reponsible for his work and clearly enjoys to study History and Biology. He has the benefit to have a pedagogically very rich home work environment (father and mother teachers) that allows him to have an unusual conception of school, in a child of his age.

Working with Logo for 9 months, his interest in Logo activities has increased surprisingly. His first encounter with Logo was trough an English version. Four months ago he insisted to collaborate in my work, testing a Logo implementation I was developing to allow young children to "plan in action", constructing procedures in "drive mode." When developping his computer work he enjoys especially the discussions we mantain. Teaching his young brother (6 years old) and presenting the discoveries to his mother, he has developped a certain sensibility to explain "what is happening" in his projects. The choice of a Logo drawing to ilustrate the cover of a written work Miguel is carrying out in the word processor, indicates that Logo means something very important to him.

Psychologically, the possibility to domain of a programming language and carry out his own projects with the computer, have clearly increased his self concept. Knowing very well that Logo is not a common language among Portuguese primary school teachers, he has got pride of his work with the computer.

The activities developed with Logo were his own projects (short games, pictures) and some work proposed by myself, as a challenge. When developing personal projects for his own and carrying out goal-directed activities, Miguel has been observed, several of our discussions have been audiotaped, and field notes and a copy of his work from the printer have been also collected for data analysis. My role as a researcher was only to clarify my understanding of Miguel' s thinking during the work with Logo, when he "thinks aloud". I encouraged the discussion about his Logo activities in situations off computer and usually Miguel started talking about it at dinner time transferring the problems to solve to common real contexts.

3. The concept of variable

It is difficult to study the variable concept isolately, being necessary to take into account several mathematical and Logo programming concepts that possibly develop in parallel with it. I will focus my attention describing Miguel' s levels of understanding of variables in the context of work with Logo.

Intuitive concept of variable. Since the beginning of his work, the attribution of a value as input in FD/BK and RT/LT commands was viewed by Miguel as a "turtle need". "It does'nt work without the value... in FD... ". In the exploration phase of his work (about three weeks) he used integers and decimals as input to the primitives, prevailing always the idea of placeholder.

Miguel:"FD needs always a value... but what are the bounds ? "

"Try to figure them out."

Miguel:"Oh!... The turtle doesn' t like 100 000 as input... but why ... if it goes out and in every time..." [referring WRAP mode].

It seems that Miguel "needs" to know how far can the turtle go, in order to choose the value to "place in the blank". This placeholder concept was reinforced when he began working with variables in Logo. Miguel' s first project was to draw several squares. His proposal was:

```
TO SQUARE
REPEAT 4 [ FD    RT 90 ]
END
```

He was waiting that in the moment the computer was doing FD he could give him the necessary information (the input). When he saw the error message "FD needs more inputs" from the computer, he introduced the number 40, in order to give the computer what it was asking for. He assumed the computer would suspend the execution of the procedure and when given the value 40 it would continue the procedure. It seems that since that moment there is a qualitative change in the placeholder idea:

Miguel: "Well... I" ve got it! I will write FD ? ... or FD something ... so the computer knows I'm going to give him a value."

__Procedural__ __concept of variable.__ The requirement to declare the variables in the naming of the procedure seems to reinforce the idea that "these variables" were different from the inputs he used with the Logo primitives. The names of the variables he used were strongly linked to their referents and to the context of the problem to solve. Despite that fact the variables were usually referred in a "quantitative" way. ":STEPS is the number of steps ... the step is the product of my procedure ... :STEPS can' t be the steps". For a long time, he never used just a single letter to define a variable.

In this *procedural* level the variable was viewed as a "name that is waiting for a value" from a set. Curiously the domain of the referents seems to reflect in the name used for the variable:

```
TO BORN :AGEYEARS
PRINT [CONGRATULATIONS! YOU WERE BORN IN]  PRINT 1986 - :AGEYEARS
END
```

The fact he has to declare the variable at the beginning of the procedure seems to be taken as some kind of information to the computer also about its domain and/or for user' s information.

"Why do you write AGEYEARS ?"

Miguel: " Ha... 19745 doesn 't mean anything...I have 11 years and a half but I was not born in19745... So you must say it...the computer doesn't know..."

When he used the variable-REPEAT command something happened

```
(1)    TO STAIR :NUM :HIGH
(2)    REPEAT :NUM [FD :HIGH RT 90 FD :HIGH LT 90]
       END
```

In (1) :NUM was assumed as a variable but in (2) the idea of placeholder was reinforced again in :NUM. Miguel tryed to explain: "I must tell how many times to repeat ... but the variable :NUM will fill the blank !?... no ... I must put it there." A simillar situation seems to occur in recursive procedures:

```
TO PINGPONG :ME :YOU
FD :ME  BK :YOU
PINGPONG :ME :ME+:YOU
END
```

":YOU is now :ME+:YOU and :ME is :ME anyway... but the :ME is another :ME ... I lost the :YOU ... just its place is there ... and waits for :ME+:YOU ... Oh...".

Miguel tryed to explain why the variable :YOU is a variable in the beginning of the procedure and then is "not so variable". It seems that a concept of dependence was coming up but that this was confunding his previous view of the variables.

Dependent and independent variables. Along the activities with the variables, Miguel was talking about them in two different ways: "My names" and "the computer names". As we discussed the Students and Professors Problem (Clement, 1980), Miguel tryed several procedures in order to explore that problem. When he writes MAKE "S 6*:P, :S was a "computer name" (a computer variable) and :P was a "my name" (my variable). The obligation to declare the variable :P was clearly linked to a kind of independent variable.

The writting form "S and :S reinforced the difference between the name and the referent (the value) but that wasn' t a very important question to Miguel. "That' s logic... the name is something but doesn 't make the person... I am Miguel but I am not D. Miguel [referring a King of Portugal]."

In this algebraic context problem, the variables were described as *unknowns* standing for "just some correct values". Implicitly there was a concept of truth domain in Miguel' s reasonning; as the variables were declared in the procedure they were viewed as "my variables" (independent ones).

4. Conclusion

The inability to deal with the notion of variable constitutes an handicap to functional reasoning that may prevent the students of recognizing and exploring relationships between variables (Ponte, 1984).

It is a difficult task to improve and model the acquisition of the variable concept by algebra beginning students. Lawler (1985) has shown that the exploration of procedures may be a powerful environment to develop mathematical concepts. When working with Logo, Miguel passed several non sequential *levels of understanding* of what a variable is. It was clear that he passed from an *intuitive* level to a *procedural* one, and these cognitive steps were strongly modeled by Logo activities.

Miguel was really doing mathematics and having the experience of mathematizing by himself. When mathematizing familiar processes is a fluent, natural and enjoyable activity, that is time to talk about mathematizing mathematical structures (Papert, 1972).

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WHAT MATHEMATICS DO CHILDREN TAKE AWAY FROM LOGO?

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This paper reports two exploratory studies aimed at investigating Papert's enquiry whether Logo may be a vehicle for fostering the growth of a 'Mathematical Way of Thinking' - "something other than algebra or geometry which, once learned, will make it easy to learn algebra and geometry." (Papert 1972, p 250).

The two studies outlined formed part of an eighteen-month classroom-based ethnographic observation of children's programming activities, which supported the view that the construction of Logo programs offers a rich (though far from unproblematic) environment in which pupils (aged 8-11) can explore a variety of mathematical themes (Noss 1985). The aim of the studies reported here was to investigate the kinds of intuitive mathematical understandings which children develop in the course of their Logo work, and the ways in which such knowledge might provide conceptual scaffolding for future mathematical learning.

Study 1 - (Geometry)

There is abundant evidence that many children's intuitive knowledge of geometrical concepts is partial and fragmented, and that it does not stand them in good stead in the context of formal geometrical problems. The question posed in the Geometry study was, does Logo make any difference?

A set of 12 pencil-and-paper test items was designed which probed pupils' conceptions of three primitive elements of each of the concepts i. length and ii. angle. The items were not aimed at measuring the children's understanding of taught geometrical ideas; rather the aim was to gain information about the kinds of geometrical knowledge which children constructed for themselves during their Logo work. The components for the concept of length were: Length Conservation (L1), Length Combination (L2), and Length Measurement (L3). For the concept of angle the components were:

Right-angle Conservation (A1), Angle Conservation (A2), and Angle Measurement (A3). The responses of 84 children who had studied Logo for about one year were compared (using a linear modelling program appropriate for exploratory data of this kind) to a comparison group of 92 pupils who had no computing experience. Details of the data collection and analysis are given in Noss (1985).

Study 1 - findings

The findings for the concept of length are exemplified by the responses on a well-known CSMS item (L1), which involves understanding that the length of a line is independent of the alignment of its end-points (Hart 1980). Here the performance of the Logo group was almost up to the level of CSMS (despite the 1-3 year age difference), in contrast to the comparison group where the facility level was some 10% lower. A similar result was obtained for the length measurement component (L3).

For the concept of angle, there were effects (significant at the 5% level) in favour of the Logo children's ability to compare unequal angles in different orientations (A2), and in their ability to identify the smallest angle in a set (A3).

When the data is analysed by sex, some surprising results emerge. For example in the length measurement category (L3), the Logo girls outscored both their male colleagues and the CSMS levels (by some 11%). For the comparison group the situation was reversed, with the boys outscoring the girls (who scored below the CSMS level). For the angle categories, the situation was still more marked, with the gap in favour of the boys in the comparison groups reversed in favour of the Logo girls for angle categories A1 and A3. In the comparison of unequal angles (A2), the significant overall effect in favour of the Logo classes was due to some 9% improvement on the part of the boys, compared with a 20% improvement on the part of the girls.

Discussion of study 1

The comparatively strong effect for the concept of angle could be explained if it is conjectured that most children have a richer store of intuitive (sometimes non-generalisable) knowledge about length than they have about angles (see Papert et al. 1979). If so, new Logo-based knowledge about lengths would be less effective in modifying existing intuitions than that concerning angle.

The theory also goes some way towards explaining the broadly convergent trend towards a differential effect in favour of the Logo girls (a similar finding was reported by Howe et al. 1982). If we can assume (somewhat contentiously) that the range of intuitive geometrical knowledge acquired by girls is often restricted compared to that of boys, then it is reasonable to suppose that the Logo activity would be more likely to displace and modify girls' existing geometrical schemas. Further research is needed on these issues.

Study 2 - (Algebra)

The second study aimed at investigating whether the experience of using algebra (embodied in Logo) could help to provide a conceptual framework for the development of elementary algebraic concepts. There were two main issues: i. How may children use the Logo ideas of a) naming and b) inputs to facilitate the conceptualisation and symbolisation of the idea of algebraic variable? ii. In what ways are children able to use their Logo-based experience as a conceptual framework for algebraic formalisation?

Four problems from Booth's (1984) algebra study were adapted so that they were appropriate for children who had not studied any formal algebra, and so that they offered pupils starting points for the construction of a relevant formalism rather than the interpretation of existing symbols. The problems were given to eight children aged between 10 and 11 who had learned Logo for some 18 months. Each child was told that the researcher worked with a group of younger children who had learned Logo; their problem was to write down solutions in the form of rules so that these younger children could understand them. Consistent with the aim of investigating what the children could learn rather than what they already knew, the children were prompted according to a loosely structured schedule (details are provided in Noss 1985).

Study 2 - findings

Consider the example of Nicola, presented with the following problem: "Peter has some marbles. Jane has some marbles. What could you write for the number of marbles Peter and Jane have altogether?"

Nicola: You could use the input again. (Note that despite the "again", Nicola has not mentioned or used inputs in the previous three problems).

Richard: Alright, show me how.

N. (writes) :Peter + :Jane = all the marbles

R: Can you read it out?

N: Peter plus Jane equals all the marbles. You use those two as the inputs, with as many marbles as you want to.

R: So what are the dots in front of Peter and Jane?

N: They're to represent that it's an intput.

R: But this isn't a Logo program is it?

N: I know, but if it was ... just to say that it's an input.

R: So what does the input actually mean there then?

N: That you can type in however size you want it or how many you want it. How ever many they want. How many they want Peter to have, and how many they want Jane to have.

Nicola's apparent conception of her variables as standing for a range of numbers does at least seem to run counter to the 'natural' tendency referred to by Booth (1984) of children to interpret variables (in the form of letters) as specific numbers.

One clue as to a possible mechanism for Nicola's view is provided by her image of the variable standing for a number 'typed in' at the keyboard (another child referred to the computer "choosing random numbers"). Such a conception only involves the consideration of values one at a time - albeit from an infinite domain - a perspective which may provide a conceptual bridge to the more generalised mathematical usage.

The rôle of Logo-based intuitions in constructing a conceptual bridge to algebra, could similarly be discerned in the formalisation process which evolved in some cases. Here are Stephen's consecutive written attempts at formalising the relationship between the number of green blocks required to 'make a bridge' over a line of red blocks (see figure 1).

1. IF :REDS = 10 [MAKE :GREENS 14]
2. MAKE :GREENS FOUR MORE THAN REDS
3. G = R. + 4

There is a progression within Stephen's three formulations which appears to represent a development from a procedural/descriptive specific-number rule (1), through a hybrid formalisation (2) in which he defines a generalised rather than specific-number relationship, to an algebraic equation (3). Is it too fanciful to view it as a chain of view-shifts in which his knowledge of Logo formalism has acted as a catalyst in the process of formalisation?

Discussion of study 2

Although this study does not readily lend itself to generalisation, it may provide pointers to ways in which using algebraic ideas in a computational setting might provide a conceptual framework for more formal algebraic learning. Such learning would not be spontaneous, at least not if the objective was conventional algebra. Nevertheless, it would be unwise to underestimate the potential of children to call on knowledge which has been generated in a different context, as the following extract illustrates.

Richard: Have you ever done anything like that before? (i.e. using a name to stand for the values of a variable)
Julie: On the other page (i.e. in a previous problem)
R: What about when you've been doing Logo?
J: Yes, when we did GAME we did it like that (GAME was Julie's project from two months earlier)
R: Can you remember what that was?
J: It did the distance round people.

Conclusion

The two studies reported here suggest that the emergence in schools of children who have developed a rich body of intuitions derived from programming may present mathematics educators with a challenging opportunity. The process by which children link disparate conceptions, both intuitive and taught, is complex and only sometimes spontaneous (see di Sessa 1983, Lawler 1985). Whether Logo-derived knowledge will make it 'easy to learn algebra and geometry' is a question which is largely dependent on further research into the conceptual linkages of which children are capable, the design of suitable learning environments, and not least to the training of teachers who are sensitive to the task.

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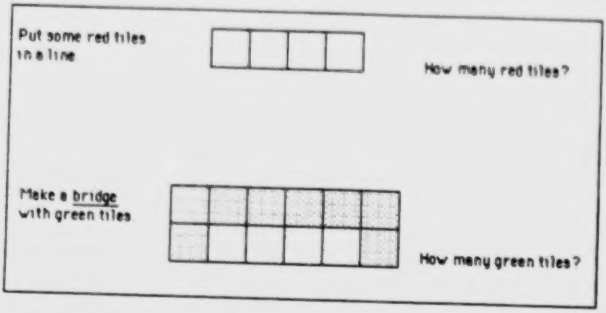

Figure 1

Initial problem card given as a starting point for problem 1 (see study 2 - Stephen's formalisations)

MENTAL REPRESENTATIONS IN PROGRAMMING, BY 15-16 YEARS OLD STUDENTS

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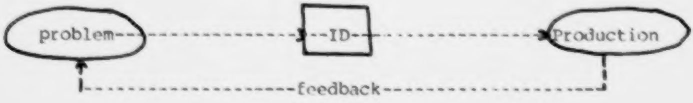
The question of mental representations constructed by students in programming is a part of a more general question: what kinds of mental representations of the informatical device (ID) interact with the cognitive activities involved in computer use for a specific purpose.

The problem is relevant to the concern of mathematical education from three points of view. First, informatics can be used as a medium and a framework in teaching and learning mathematics. The specific properties of the ID, the informatical constraints can induce specific effects. Second, informatics as a specific field of knowledge is partly related to mathematics: problem-solving in these two fields present differences and also invariants (Rogalski, 1985b). Concepts like variables, iteration are involved in each field: the possible interactions between the representations students get on these concepts are important in learning mathematics as well as in learning informatics (Samurcay, 1985a, 1985b). Third, informatics can be used as an aid to solve mathematical problems. Today, this question does not concern a wide part of mathematical activities by students even at a high level; nevertheless we have to anticipate the kind of psychological, cognitive questions issued from the development of an informatical environment in mathematical problem-solving.

What can produce the representations on the ID when informatics is used as a tool, as a framework, as a scientific content?

The general situation in using an informatical device can be represent by the following schema:



In such a situation, the student has to lead two kinds of cognitive activities in order to control his/her production, by respect to the target state researched for the given problem: first, a "logical" analysis of the production in terms of the content of the problem (it can be a mathematical one, or an informatical one, in the case of programming, for instance), second s/he must take into account (even unconsciously) the properties of the ID which played the role of a medium.

In fact, the properties of this medium may play a productive as a reductive role. We have presented this notion of productive and reductive role from a work with A.Robert on graphical representations used in high school (Rogalski, 1984).

The reductive features are limiting the relationships that can be represented by the medium (as a tool and as a signifiant): for instance infinity cannot be represented: all processes as all numbers are discretised, just as in graphical representations; more:the internal representation of numbers in the ID introduce strong reductive specificities.

The productive features allow the student to "see" properties s/he has not introduced her/himself in the device. Let us take for instance a software presenting a task :"reach the target" in order to study the relationships between number and space used by young pupils (Rogalski, 1985a). When the child enters a number, s/he can see an immediate effect on the screen quite different of her/his proper action: a trace appears on a line, and the child has the possibility to appreciate the distance between this trace and the target.

Computer-use, at all levels, introduces such productive and reductive features: knowing them appears me to be important to master the use of informatics as a tool or an object for teaching and learning. But there are another cognitive questions for the use of informatics: the mediation between problem and production, schematised in the schema above, involves a technical device, with a high level of complexity, which has the status of a "black box". This "black box" is not a neutral medium: hard- and software make it reactiv, in a dynamical way, to the proper actions of the subject using it. What kind of representations may induce such a reactivity? What kind of hypothesis about the meaning of the "reactions" of the ID? What kind of invariants concerning the functional properties of this ID? In some way, the use of a computer introduce a situation of communication between the human subject and the informatical device: what happens when they are speaking maths?

Mental representations about the informatical device and concepts acquired in computer literay

Our work hypothesis is that programming can be taken as a model for studying the mental representations about the ID. More precisely, computer literacy allows to observe the beginning of a process. At this moment the conceptual notions in the field of informatics are only the basic schemas for sequencement, conditional structures, iterative structures, and the simplest data structures. The domain of the problem to be programmed are also simple domains, which can be "hand-solved" by the students. The fonctional properties of the ID which interact directly with the activity of programmation are only a part of the whole system.

The analyses of the cognitive activities of beginners showed that the acquisition of the rules on computer functioning and the progressive integration of the control structures lead the beginner to use a strategy of mental execution when writing programs (Hoc,1983). They also showed the role of mental representations in learning a programming language (Hoc, 1977).

At the level of computer literacy, many -and purhaps most- of the informatical concepts have mathematical concepts as precursors; it is the case for the notion of variable, for the conditionals, for the algorithms themselves. The passage from the mathematical conceptual fields in which they are organised to informatical conceptual fields is a complex process, involving central properties of the ID (Rogalski & Samürçay).

So, the construction of mental representations of the elementary informatical concepts implies representations of what "knows" the system and how it "knows" and "acts". The sequencement of actions (instructions in the program), the management of the inputs and outputs are two primary questions in all procedural languages. In fact they are necessary in the passage from a mental execution to a real running of a program on the ID.

Mental representations and programming

In writing a program the student must get a representation of the functional relationships between the written program, the computer on which it will be executed and the operator who will use the program and in particular who masters the input data.

Wrong or incomplete representations will interfer with the process of algorithmisation. In all cases the properties of the representations play a role in the meaning that the students attributes to the result of a given execution.

Preliminary observations of 15-16 years old students programming with PASCAL showed a wide variety of questions of representations and attest the difficulty of an experimental study. Most of the representations are unconscious and non explicitable by the students; they interact with their productions at a very deep level.

The existence and the properties of the mental representations can be attested essentially by analysing their spontaneous behavior, and not by using responses to specific sollicitations on these points. This allows to begin to constitute a set of "critical incidents" to be researched in the programming activities of the students.

As an example, we can give the impossibility to analyse a "prompt" on the screen as the fact that the system is waiting an input if the student trying his/her program does not wanted an input at this point (alghough s/he wrote an input instruction in the text of the program corresponding at this moment of execution). Another example of the same order is the "hand" execution of what the student wanted to obtain and not of what s/he has written.

Nevertheless we organised "local" experiments on the representations about the relationships between the program, the operator and the computer in class-room situations led by the teacher during a normal curse on informatics (10th grade students, 1h30 or 2 hours sequences each week).

We collected individual responses of the students on three problems presented after 15-20 hours of programmation (with the french teaching language LSE, near of BASIC by many respects).

Problem A: at the beginning of a program students had to introduce an if...then...else... structure just after the input of data in order to control the validity of these data (hours, minutes and secondes). The structure they had learned in LSE was the GOTO statement (in LSE: aller en). The relevant points are the following: does the student use a GOTO (instruction to the computer), does s/he write a new input instruction if the data are incorrect, does s/he use an instruction like WRITE" GOTO.." (order to the operator).

Problem B: the student has to write a part of a program asking to the operator if he/she wants to stop here or to continue (with a yes/no response to the question "do you want the program stops here) and using an appropriate if...then...else structure according the instructions to be performed to the response of the operator, that is the instruction TERMINER in one case and nothing specific or a PRINT instruction informing the operator of the pursuite of the program preceeding the rest of the program. The relevant points are the following: does the student write the good instruction TERMINER if the response is YES, does h/he write the instruction PRINT "TERMINER" instead of the preceeding one, does h/he write an instruction TERMINER after a NO response with eventually an "informative" PRINT instruction like PRINT "we continue".

Problem C: a written program is presented to the students, in which a question "do you want another running? yes/no", an the instructions conditionned by the NO response; the students had to predict what will happen if the operator enters an irrelevant response (like: why not?).

Conclusion

This study of representations on the ID in programming shows that in the beginning of the acquisitions two level of representations can be distinguished: at the first level students confuse the function of execution of the program by the computer and the execution of actions by the operator, at the second level they attribute to the computer some of the semantic capacities of the human operator. There is a hierarchy between these levels: the second one seems to have a longer life than the first one. These wrong representations interact with an insuffisant representation of the sequentiality of execution, and affect the production of a part of the students. Others recents observations show that the confusion between the role of the operator and the role of the computer remains for some students after 40 hours of programming in strong interaction with the difficulty to construct an adequate representation of the assessment instruction. More research is needed if we want to master the development of mathematical algorithmics activities in relationship to programming, and more generally computer as an efficient tool for maths education.

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LEARNING RECURSIVE CALLS IN BUILDING UP LOGO PROCEDURES.

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There is now an important literature devoted to the interest of LOGO and LOGO based environments on mathematics learning and teaching. But apart from the fact that we know very little about the precise conditions in which LOGO is really interesting for the study of mathematical concepts, we also know very little about the difficulties of conceptualisation of computer science concepts through LOGO. We are going to give indications about some problems we have met in the study of programming linear recursive calls in LOGO.

1. ITERATION AND RECURSION IN MATHEMATICS AND COMPUTER SCIENCE.

1.1. Iteration and recursion are basic notions in modern mathematics. They are theoretical fields of study and practical tools for calculation and representation in elementary analysis, arithmetic, computing algebra (Hörner algorithm), geometry. In former teaching of mathematics, the lack of appropriate device to perform effective calculation, lessen the practical interest for iteration and recursion, in spite of the fact that recursion for instance was important in the proof by recurrence process. Now, through the means of programming, interest increased, in the teaching of mathematics, to define new objects and tools to work with, associated with the power of computer.

1.2. In computer science and programming, iteration and recursion play a central role. The organisation of repetitive and iterative processes has different aspects in various sorts of languages. In this field of interest we can define two models :

- The pure iterative model : *while ... do* in Pascal and associates: *Repeat ... until*, *for ... next* with this kind of loop, actions are organised according to a sequential pattern.
- The recursive model of Lisp and LOGO, where the actions (or relations) are planified (or described) according to a part-to-the-whole pattern.

1.3. Previous work has been done about the way students understand loop and recursive structures. Soloway and others developed a rather deep analysis of planification during the construction of *while ... do* loop in Pascal. Pea and Kurland pointed out the difference, from the point of view of the student, between tail recursion and full recursion, and the difficulty to interpret the second one. To deepen our knowledge of conceptual difficulties it was necessary to identify students' models through a wide variety of tasks about programming.

2. LEARNING LINEAR RECURSIVE CALLS IN LOGO.

2.1. In France, Computer Science is not a part of compulsory education at the secondary level. So there are no traditions and no teaching problems ; no more than there is any general computer literacy to interact with general trends of teaching and general conception of pupils.

The only way to understand and analyze conceptual difficulties is the way of teaching the content of interest for the researcher.

2.2. So, in the field of programming in LOGO, our main object of study was the relation between a learning process and the conceptions of children. These depend on learning, but they depend also on the proper way in which children conceptualize the topic.

We are sure that various learning processes can be chosen and described and we hope that it will be possible, in a while to have a good description of invariants in children models. For the moment, the only way of access we have is the study of effects of particular learning processes.

2.3. Formally the general structure of linear recursive call (one call in the program) is the following in LOGO : we call it central recursive call.

```
TO      < name-of-procedure>     < list1 of variables>
     IF (cond-of-one-variable)   [<external-procedure-4> STOP]
   <external-procedure-2>   <list2 of variables>
   <name-of-procedure >     <list1 of modified variables>
   <external-procedure3>    <list3 of variables >
END
```

When external-procedure2 does not exist, we have the classical tail-recursion

When external-procedure3 does not exist, we have the classical full-recursion.

Going on in the identification of levels of difficulties with recursiv calls in all these aspects implies an analysis of the way children interpret th main features of these programs : relations between external procedures an recursive call, evolution of variables, testing, etc ... and their correspondanc with self reference and nesting.

2.4. The understanding of this structure, in general as well as in the case of its application to a particular field has to be testified through various kinds of behaviors :

- write a procedure to product a defined object
- predict what object to be producted by such a procedure
- explain how this procedure actually runs
- modify a given procedure to fill out additional conditions.
- .....

3. DESCRIPTION OF THE BUTTERFLY SESSION.

3.1. Our pilot study was conducted with pupils of grades K8, K9 (two last years of french secondary school : college). The teaching period was approximatly 30 hours. Pupils worked by pair on microcomputers; one hour a week.

The general purpose of the teaching sequence was to develop the various ways of using and building up linear recursive call structure in the field of LOGO-graphics and schemas of monotones sequences in N. During the learning process, exercises solved by pupils were in correspondence with the tasks listed in 2.4.

3.2. Learning through problems has special aspects in the case of computer science depending on the object-for-learning which is selected to be taught. Apart from the general structure quoted above, pupils have to identify all kinds of relations between elements of this structure and elements of the figure, between running the program and obtaining the desired effect, between modifying the program and special graphical conditions, etc...

For instance, if we want to focus on the place of external procedures in the general structure and the order in which the elements of the figure are drawn we can use a strategy like the following one.

3.3. These activities are proposed with the general objective of identifying central recursive call as a means of solving a certain type of problems. At this moment, pupils have worked only with a full-recursion structure.

A. Give a program to draw the following nested hexagons.

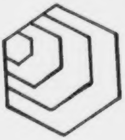

Fig. 1.

B. Give an equivalence of this program in terms of the sequence of calls to **HEXAR** where hexar is the program

```
TO HEXAR :C
     REPEAT 6  RIGHT 60  FD :C
END
```

C. What is the drawing product by the following procedure **QUIZZ** when :N = 5

```
TO QUIZZ :N
     IF :N = 0  STOP
     HEXAR :N * 5
     QUIZZ :N - 1
END
```

This was the meeting of the students with tail recursion.

D. What about the order of the two sequences of hexar in the first program **(A)**.

E. Write the same programs with **HEXAL** in place of hexar :

```
TO HEXAL :C
     REPEAT 6  LEFT 60  FD :C
END
```

F. Write a program to draw the following butterfly in the order indicated below :

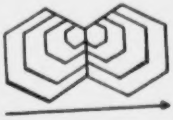

Fig. 2.

G. How can we put antennas to the butterfly in the same procedure.

4. INVESTIGATING STUDENTS MODELS ABOUT CENTRAL RECURSIVE CALL.

4.1 As we pointed in 2.4., various tasks must be given from the point of view of learning as well as from the point of view of investigating pupils conceptions. For instance some important facts about the running of central recursive calls concern :

A. The sequence of calls to external procedures, mainly their order in relation to the law of evolution of the main variable of the program.
Two models may be indentified :

G.B. *Global model which states that in the case of a decreasing law of evolution n   n - 1, external-procedure-2 (which is before the recursive call) is called in decreasing order, while external-procedure-3 (which is after the recursive call) is called in increasing order.*

A.M. *Analytical model which consist in simulating the running of the program. For instance, if we call for BUTTERFLY 5, beginning of A.M. is 5 > 0 then HEXAG 5 * 10 .....*

The result is the same, but the representations of the pupils are significantly different.

B. The place of call to external procedure-4 in the sequence.

C. The modifications to be made so that the main procedure obtains a different drawing.

4.2. We don't argue that these three aspects cover all what is needed to define the understanding of recursive call. For instance we didn't take into account in this analysis the fundamental phenomenon of nesting which concerns the main procedure itself. Identification of nesting was the objective of another teaching session with the general philosophy of defining a modification of the program for which the running leads to the displaying of the sequence of recursive calls : trace system.

4.3. A questionnaire concerning the three aspects of 4.1. was given to the students after the butterfly session, with the objective of indentifying the exact models students were using. The basic figures were the following ones :

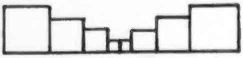

Fig. 3.

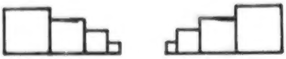

Fig. 4.

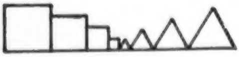

Fig. 5.

We cannot give here a detailed analysis of the results, so we will take them in an indicative sense.

First question, about the production of the figure (knowing the program) and the sequence of graphical instructions : (figure 3)

12 students out of 24 used G.B. (8 correct. 4 erroneous).

8 students out of 24 used .M. (4 correct. 4 erroneous).

4 were completely wrong.

Second question, about the effect of a SKIP procedure in the precedent program, in terms of drawing as well as in terms of sequence of graphical instructions : (figure 4)

3 students out of 24 put SKIP at the correct place, among them

7 (out of 8) were correct with GB, 2 (owt of 4) were correct with A.

Third question : there were two options : (figure 5)

A.Given a modified program, propose the corresponding modified figure.

B.Given the figure, propose the corresponding modified program.

In the case of A, pupils who succeeded in the first question succeeded in the third (which seems normal).

In the case of B, 6 out of 7 succeeded.

5. CONCLUSION.

The complete interpretation of running linear recursive calls suppose a good understanding of self-reference and nesting. It seems to us that the analytical model, which seems too formal, is not sufficient. The previous study has shown the importance of a general rule like the Global Model. To complete this model in the direction of understanding the effect and the running of self-reference and nesting, it is necessary to give tasks about successive calls of the program itself for instance in programming a trace system.

INITIAL REPRESENTATIONS OF STUDENTS IN USING RECURSIVE LOGO PROCEDURES

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Abstract 26 eight graders (13 - 14 years old) were observed as they solved two different types of task in which they have to use the notion of linear recursive call The production of students are analyzed in termes of the representations they have constructed during the teaching experiment conducted with them in classroom situations

1. Introduction

The recursion is considered as an important concept that people have to learn. The work on recursive procedures in programming activity is seen as a possibility to understand better some mathematical concepts, for instance, concept of recursive functions (Hausmann, 1985). The implicit hypothesis is that the programming will constitue a privilegied domain in which student will be able to experiment about recursion before construct a formal knowledge about it. But how do people learn this complex concept? Which kind of cognitive difficulties they encounter? In which kind of situations it is possible to teach the notion of recursivity?

The study we present here is integrated in a more large didactical research on learning recursive procedures by middleschool students. It concerns a very early moment in the acquisition process and attempts to analyse how this new knowledge is used by students in problem solving.

First we distinguish between the recursive procedures corresponding to recursively defined objects (nested or fractal objects) and those corresponding to iteratively structured objets (par exemple, embedded squares). In this last case we'll talk about linear recursive call. The general structure of this kind of procedures used to program iterative drawings in LOGO is :

```
TO PROCEDURE : (list of variables)
IF (end control on 1 variable) (P1 STOP)
P2 :(list of variables)
PROCEDURE :(list of modified variables)
P3 : (list of variables)
END
```

P1, P2 and P3 are external procedures. This present form is a **central recursion**. If P2 is empty we obtain a **full-recursion**, if P3 is empty we obtain a **tail-recursion**.

In the litterature, the studies on learning recursion concern most often the understanding (Pea & Kurland, 1983) or the construction

(Pirolli & Anderson, 1985) of the tail recursion. This form is considered more easy to learn for the student than the others. In the didactical experiment we have constructed, the students encountered first the full recursion form, and then the tail recursion and the central recursion. The problem solving session we have observed will allow to analyse the effects of this first learned model on the productions of students.

2. Population and method

26 eight grade college students are individually interviewed during 30 mnts. The production task is proposed to 15 students (4 of them worked in group of two, so there is 13 protocols analyzed). The comprehension task is proposed to 11 students. The interviews are recorded and students' work papers are collected. The protocols are analyzed as regard to the criterion we define in below.

3. Task analysis

Two kinds of task are proposed to students

i.Program production task: given the figure.1 it is asked to students to write a procedure to obtain it. If the students were not produce spontaneously a recursive procedure, it is asked explicitely to write it on.

```
TO STICK : length
FD :length BK: length
END

TO JUMP
PU RT 90 FD 10 LT 90 PD

TO FIG : n
IF :n = 0 (STOP)
FIG :n-1
JUMP STICK : n* 10
END
```

(a plausible solution)

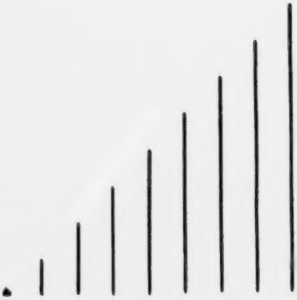


Figure.1

The obtention of the figure doesn't necessitate the use of recursive call, it can be obtained also by using a step by step procedure. We assume that spontaneously the students will use this kind of procedure and try to transforme it to a recursive one, with the aim to make it short. Three types of procedures can be used:

-SSP: step by step procedure (direct mode);

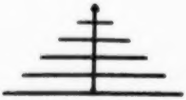
-MP: moduler procedure in which the figure is analyzed as composed by two sub-procedures: a parametrized procedure for sticks, and an interface procedure to move the turtle to one stick to another.

-RP: procedure with recursive call in which the figure is analyzed in terms of invariant relations between parts and whole.

2. Program comprehension task: given the following LOGO procedure, it is asked to students to draw the figure obtained when the procedure TRUC 5 is called. As a second question it is asked to write on the list of graphical instructions in order in which they will be executed when the TRUC 3 is called.

```
TO STICK: 1
FD :1 BK :1
END

TO TRUC :n
IF :n =0 (STOP)
TRUC :n-1
BK 5
LT 90 STICK :n*10 RT 180 STICK :n*10 LT 90
END
```



(expected solution)

According to the level of conceptualization, this recursive procedure can be read differently. We expect that many of students use or a linear model in which the iterative aspect is neglected, either a simple repetitive model in which the property of embedding is neglected.

We assume that this two tasks (production and comprehension) don't involve the same kind of activity. In the production task, the students have more than one possible solutions to obtain the given figure even if they are not recursive, and the solution may progress during the problem solving session. While in the comprehension task the number of possible exact solutions are restricted: there is one possible draw corresponding to the given procedure.

4. Protocol Analysis

1.Production task: Protocols are analyzed not only as regard to their correctness but also in terms of evolution during the interview session.The results may be summarized as in table.1.

SSP	MP	*	RC
4	4		
	6	5	1
			3

table.1 (N=13)
(*the students who received help from the observer)

As seen on the table.1, 4 students start by a step by step procedures and construct then modular procedure; 6 students start by a modular procedure: 5 of them construct recursive procedures with observer's help, 1 only do it without help; 3 students start by recursive procedures without passing by intermediary steps.

The analysis of the verbal protocols allow to notice the following points:

* The idea of recursivity is evoked by reference to the form of the control line: "**Ah, I see, the thing with if equal and step....**". However this evocation is quite formal, because the students are not always able to identify the variable on which they have to construct the end-control and the particular value of this variable has to take.

*Some of the students make reference to the line of of recursive call. They have also big difficulties to identify the new value of the recursive call variable, which describes a relation of embedding between the parts forming the whole. This idea is explained by one of the students: "**I don't know if ! have to put -1 or +1,...it depends,...this is something which makes the same thing getting bigger or getting smaller**".

*For some students a recursive procedure is a technic to make short a modular procedure. By using this property, the recursion is assimilated to the repeat instruction. This interpretation creates a difficulty in the program production, because when the sticks are structurally identical their length is varying. Some of the students proposed to add one more variable to change the length in each execution of repeat instruction.

2.Comprehension task: Two criterion are used to analyze the students' productions. The first one is the model used in the reading of procedure. As we've noticed above, two kinds of interpretation are possible:

*linear model in which the iterative aspect is neglected. In this interpretation the students execute each line of the procedure line by line and stop at the end of the procedure without return back. We consider that in this model only syntactic features are read.

*the second model corresponds to an iterative model . The students "know" that there is more than one execution of the lines. We consider that in this model, a semantic understanding is raising.

The second criterion concerns the interpretation of the recursive call line. There is two different interpretations:

*this line is ignored, because the student have difficulty to assign a meaning. Then the execution is realized by the first call-value

*the recursive call line is taken into account; the value in the first execution is the call value -1.

By combining these two criterion, we obtain 4 different cathegories (see table.2). As shown by table.2, the majority of the students (7/11) read the procedure linearly, only 1/3 of students saw the idea of iteration in the recursive form. However the full meaning is not yet constructed, because the scheme used is a tail recursion scheme.

	linear	iterative
call line ignored	2	4
call line read	2	3

table.2 (N=11)

As shown by the table.2, 3 students gave a correct recursive interpretation; 4 of them gave an iterative interpretation (without the property of embedding); 4 gave a linear interpretation.

5.Conluding remarks

As seen by our results the production and the understanding of recursive procedures are not an easy task. In the production task, we see that the modular procedures concern the majority of the observed protocols. Students evoke the idea of iteration but they have still big difficulties to transforme the regularities they observe between the sub-procedures, into a recursive writing.

In the comprehension task, also the correct interpretation of the full recursion form concerns a very restricted number of students. The majority interprets the full recursion schema as a linear procedure or as a "go to up" schema which corresponds to the manner that the tail recursion can be read. They have difficulty to represent the order of execution and the idea of embedded recursive calls. The inverse relation between the ordre of recursive calls and the order of execution is very hard to understand for the students for whom the actual model is the "familiar actions" model.

It is interesting to notice that this schema appears only in the comprehension task. While in the production task, none of the students use this scheme which is not learned.

We conclude with the hypothesis that if the tail recursion sheme is easy to understand and to construct, because it is very close to the familiar procedures of students. We argue that this scheme which is based on the mental execution of the procedures, constitues a high-level obstacle for the construction by students on analytic scheme which is necessary to the conceptualization of the recursion as a concept.

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LOGO AS A CONTEXT FOR LEARNING ABOUT VARIABLE

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Introduction

For many pupils algebra forms a barrier to the understanding and enjoyment of secondary school mathematics. Research into children's understanding of algebra has highlighted the problems children have with interpreting the meaning of letters and with formalising and symbolising a generalisable method (Kuchemann 1981, Booth 1984). Given this background of children's misconceptions, it seems possible that using variable in a programming context could form a conceptual framework for the use of variable in a non programming 'paper and pencil' context. If computer programming does offer this potential we believe that the language to use for this purpose is Logo for reasons we have detailed elsewhere (Holyes, Sutherland & Evans 1985). In an exploratory study with eight pupils (aged 10+) who had learned Logo for eighteen months Noss (1985) reports that the "children perceived a variable as standing for a range of numbers, a finding which contrasts with the natural tendency (Booth 1984) of children to view variables in mathematics as standing for a specific value". Other research suggests that the use of variables in Logo is not likely to occur 'naturally'. Hillel working with nine year old children reports "that aside from difficulties in defining a general procedure there is more basically a lack of an immediate sense of the necessity (our emphasis) to define such procedures". (Hillel, Samurcay 1985)

As part of the Logo Maths Project, research is being carried out to:

- trace the development of understanding and use of the concept of variable in a Logo programming context by reference to the work of four case study pairs of pupils during their first three years of secondary schooling. (11-14 years)
- relate the pupils understanding of the concept of variable in Logo programming to their understanding of variable in traditional 'paper and pencil' algebra.
- develop and test out materials designed to aid the 'transfer' of the conception of variable derived within a Logo context to a non Logo context.

The Concept of Variable in Logo

A useful definition of variable in Logo is given by Hillel and Samurcay: "From a cognitive psychology viewpoint, the concept of variable represents , as do the concepts of iteration and recursion, an invariant. This invariance is characterised by the attribution of a name to the variable and by the control of its value." (Hillel, Samurcay 1985a). By carrying out both an *a priori* and an ongoing analysis of the situations in which children use variable in Logo categories of variable use have been identified. They provide a framework for analysing the pupils' understandings:-

- One variable input to a procedure (not operated on within the procedure).
- Variable input as scale factor (to distance or angle).

  This type of variable input is often used by pupils as a way of generalising a fixed procedure (see for example Fig 1.a.)
- More than one variable input to a procedure.

  Pupils often use more than one input to avoid expressing general a relationship between variables within a procedure. (see for example Fig1b)
- Variable input operated on within a procedure.

  In this category any general relationship between variables within a procedure is made explicit by operating on a variable input to express an internal ratio. (see for example Fig. 1c).
- Variable input to a general superprocedure which calls a general subprocedure alongside. (This includes recursive calls).
- Variable used within an assignment statement (i.e. MAKE).

  Because of the problems with global variables the case study pupils have not yet been introduced to the MAKE statement.
- Variable input to define a mathematical function in Logo.

  In this category variable is input to a procedure, which acts like a mathematical function, that is it is operated on within the procedure and the result output from the procedure to be used by another Logo function or command. (See Fig. 3a)

```
TOM :SCALE
LT 90
PU
BK 90
PD
FD MUL :SCALE 60
LT 45
FD MUL :SCALE 20
RT 90
FD MUL :SCALE 20
RT 90
FD MUL :SCALE 20
RT 90
FD MUL :SCALE 20
END
```

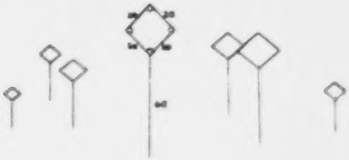
Fig. 1a (see Footnote 2)

```
KITE :YT :HT
RT 45
FD :YT
RT 90
FD :YT
RT 90
FD :YT
RT 90
FD :YT
BD :YT
RT 90
FD :YT
RT 45
FD :HT
END
```

Fig.1b

```
SQUAH :NUM
LT 135
REPEAT 4[FD :NUM RT 90]
LT 135
FD MUL :NUM 3
END
```

Fig 1c



In Logo variables are used as part of procedure definitions. Although not the focus of this paper the issues of subprocedure, modularity and sequencing are related to variable.

The Need for Teacher Intervention.

It is obvious that pupils must be using variable in their Logo programming before there is any possibility of the Logo experience enhancing their learning of variable in 'traditional' algebra. Analysis of the first eighteen months of transcript data indicates that, first, pupils rarely chose projects which 'needed' the concept of variable and, second, that even when we perceived a need for variable in a pupils' project or in a 'teacher-given' task, and intervened appropriately, the pupils were resistant to using it. This was the case for pupils with both little or no experience of variable in 'traditional' algebra. It was decided therefore to introduce the concept of variable to all the pupils within a structured task. This structured task was aimed at provoking the pupils to USE the concept (a first stage of the USING, DISCRIMINATING, GENERALISING and SYNTHESISING model. (Hoyles 1986)). Analysis of the data also indicated that pupils found working on goals which involved designs with letters to be motivating. The following task was therefore devised:

Variable Letter Task

The pupils are given a fixed procedure to draw the letter L. They are then shown how to change their fixed L procedure to a general L procedure by multiplying each distance command by a variable. They are encouraged to make sense of this new general procedure for the letter L by trying out a range of inputs; to use decimal input in the context of being asked to draw the smallest and the biggest possible L; and to explore negative input. They are then asked to define a general procedure for several letters of their choice and to use all these letters to build up a design on the screen.

The pupils found this task very motivating and extended the task to produce a range of variable letter designs (Fig. 2.)

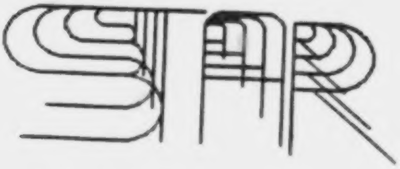


Fig. 2. Extension of Variable Letter Task.

Over the next twelve months the case study pupils were given a range of structured tasks for which variable could be used as a tool for solution. They also worked on projects of their own choice during this period. One aim ofthe research was to relate the pupils' understanding of variable in Logo to their understanding of variable in traditional 'paper and pencil' algebra and to develop materials which aid transfer between the two. It was decided to base these materials on the isomorphism between using variable to define a funtion in Logo and using variable to define a function in 'traditional' algebra. It was hypothesised that helping pupils to make the link between these two contexts would provide the basis for the transfer of variable from Logo to algebra. The following is a brief outline of the function material:

Function Material

Pupils are shown how to define a simple mathematical function in Logo. (Fig. 3a).

```
ADDFOUR :X
OUTPUT :X + 4
END
```

Fig. 3a

IN	OUT
3 ->	7
-2->	2
1.5->	5.5

Fig. 3b

$F(X) = X + 4$

or $X \rightarrow X + 4$

Fig. 3c

They are then encouraged to experiment with a range of inputs (including decimal and negative numbers (e.g. by typing PRINT ADDFOUR 3, PRINT ADDFOUR 1.7, PRINT ADDFOUR -2.5) and to define other functions using the operations of subtraction, division and multiplication. When they are able to define simple functions, they are given a game which involves one pupil defining a function and the other pupil guessing the function by trying out a range of inputs. It is an important part of the process that the pupil guessing is asked to build a mapping diagram as a tool to help work out the function. (See Fig. 3b). For some pupils the material is extended to include ideas of inverse and composite function, with the guessing game being a motivating part of all the new material. When pupils are confident about defining functions in Logo they are given 'paper and pencil' tasks away from the computer to provoke them to make the links between function as represented by a Logo procedure, function as represented by a mapping diagram (Fig. 3b) and functions as represented by 'traditional algebra' notation (Fig. 3c).

Discussion of Pupils' Understanding of Variable in a Logo context.

Ongoing analysis of the transcript data(after 60 hours of "hands on" Logo programming), structured interview data and specific tasks given to the pupils during a 'research day' at the Institute of Education is highlighting the following issues:

The Significance of the Naming of the Variable.

The case study pupils were introduced initially to meaningful variable names (e.g. SCALE, SIDE, NUM). The transcript data and pupil interviews showed that pupils were attaching undue significance to the naming of the variable. It was decided to intervene specifically to show pupils that they could use any name and the pupils were encouraged to make up 'nonsense' names. In addition the function materials used a range of variable names including single letter names. The majority of pupils seem to have passed through the stages of using meaningful names, to choosing nonsense names and then returning to meaningful names (e.g. SIDE; SIDE2) or abstract short names (e.g. X,A).

The Meaning of the Variable Name.

Discussion with the pupils about the meaning of a variable name elicits responses of the form "the size of what it is going to be", "SIDE stands for how far you want to get it to go", "SCALE lets you know it can make it as big or as small as you want it". The pupils however seem to bring their mathematical under- standing of number to the Logo situation. This means that the pupils' idea of 'range of number' can be restricted (e.g. to positive integers). At least half of the case study pupils were resistant to using decimals. In order to extend the pupils' understanding of 'any number' structured tasks will need to be devised. (Variable input as scale factor e.g. can provoke the use of decimal input).

The Critical Nature of the Variable Letter Task

None of the case study pupils were motivated to take on the idea of variable in Logo before they were given the variable letter task. It is hypothesised that the idea of changing a fixed procedure to a general procedure by scaling distance commands is conceptually easier for pupils to use than making a general relationship explicit by operatng on a variable input to express an internal ratio. Pupils know that using scaling input will produce variable screen output but are not necessarily aware of what exactly is varying. Four months after the introduction of this variable letter task three out of eight pupils still used this technique to draw Fig. 1. Three out of eight pupils perceived the internal ratio, defined one variable and operated on it (see for example Fig. 1c). Two pupils defined two separate inputs without taking into account the internal ratio (see for example Fig. 1b).

Conclusion

Our research supports the findings of Hillel and Samurcay(1985) that without specific teacher intervention pupils will not use variable in their Logo programming projects. Although the variable letter task has been successful in encouraging pupils to begin to use variable in Logo, the research highlights the need for pupils to experience variable in many different situations before a synthesis can take place. Structured interviews have still to be administered in order to probe the pupils' understanding of variable both in Logo and in a 'traditional' algebra context, and to evaluate the 'transfer' materials. The results of these will be reported at the conference.

Footnote 1.

The Logo Maths Project which commenced in Sept. 1983 is monitoring and evaluating how Logo can be used within mathematics classrooms. Two computers are placed in the classroom and pairs of pupils take turns to work with Logo during their 'normal' mathematics lessons. The researchers act as participant observers. Systematic data is being collected throughout the three years of the project for four pairs of case study pupils (aged 11-14), one boy pair, one girl pair, and two mixed pairs. Pairs where chosen taking into account spread of mathematical attainment and the teachers"s opinions as to constructive working partnerships. The data includes recordings of the pupils' Logo work, all the spoken language of the pupils while working with Logo (a video recorder is connected between the computer and the monitor), the researchers interventions and a record of all the other mathematical work undertaken by the pupils. The video recordings are transcribed and these together with researcher observations and teacher and pupil interviews provide the basis for the research results. Since 1984 the research has been extended into ten further classrooms where the control of the Logo work is the responsibility of the teacher rather than the researchers.

Footnote 2.

The case study pupils worked with RML Logo which does not possess infix arithmetic operations. The procedures in this paper are given in Apple Logo with the additional prefix arithmetic operations ADD, SUB, MUL, DIV.

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The Development of Concepts and Skills Related to Computational Estimation

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The problem: Recent documents reviewing U.S. school mathematics programs and recommending change challenge mathematics educators to increase the emphasis on computational estimation (NSB Commission, 1982; NCTM, 1980). Some of the concern was derived from calculator usage and a need to find gross estimates in order to evaluate input/output data. A more basic justification was offered by Edwards (1984) who argued that computational estimation leads to number sense--a "feel for numbers".

Compliance with these recommendations is difficult. Computation estimation, a complex interplay or concepts and skills, has been defined by Reys, Rybolt, Bestgen, and Wyatt (1980): "The interaction and/or combination of mental computation, number concepts, technical arithmetic skills including rounding, place value, and less straight-forward processes such as mental compensation that rapidly and consistently result in answers that are reasonably close to a correctly computed result. This process is done internally, without the external use of a calculating or recording tool" (page 6). Case, in applying his model of intellectual development (1985) to computational estimation (1986), hypothesized that although young school-age children can perform prerequisite tasks, they do not show real understanding of simple estimation tasks until about age 12. Futhermore, it is not until adolescence that students become capable of computational estimation in the Reys et al sense.

The study. The purpose of the study, partially described here was (1) to identify factors necessary for proficiency in computational estimation, and (2) to investigate the development of these factors. In the first phase of the study, children in grades 2, 4, 6, 8 and 10 were tested collectively and individually on estimation problems in a "store" setting, and on number sense problems including comparison of numbers, use of powers of ten, and judging relative size of numbers. The testing phase provided the groundwork for subsequent hypotheses testing. A process-product computational estimation model which included factors involving concepts, direct and indirect skills, and attitudes was formulated (see Figure 1).

For the second phase of the study, test items consistent with data collected in the first phase and with Case's theory of intellectual development were designed. These grade-appropriate items covered the factors identified in Figure 1. At each of grades 3, 5, 7 and 9, three teachers completed individual student profiles. Twelve students at each of the three grade levels who ranked in the middle half of their classes in mathematics, who portrayed interest in mathematics, and who were reflective and willing to explain answers were selected to be individually interviewed.

Procedures and results. During the 45-60 minute audio-taped interviews, students were initially questioned on mental computation and other related skills as delineated in Part III of Figure 1. The bulk of the interview was devoted to items covering Parts I and II. Approximately half of these items, direct or D-problems, consisted of a description of a situation calling for computational estimation followed by solution-explanations of two or three hypothetical students. Interviewees were questioned about the acceptability of the hypothetical explanations and the resulting estimations and requested to contrast the explanations. For the other half of the items, open-ended or O-problems, students were asked to select and describe their efforts when solving a series of computational estimation problems. Finally, attitudinal data to augment teacher profiles (Part IV of Figure 1) was gathered.

The remainder of this paper will be restricted to a limited analysis of several D-problem tasks measuring the conceptual and skill level factors identified

A BREAKDOWN OF FACTORS RELATED TO COMPUTATIONAL ESTIMATION

I. CONCEPTUAL LEVEL FACTORS

A. Process Factors

1. Recognition that there is more than one appropriate process of obtaining a computational estimate.

2. Recognition that estimation processes involve computing with approximate numbers.

3. Recognition that some processes are more appropriate than others, depending upon the type of problem and degree of accuracy demanded by the problem.

B. Product Factors

1. Recognition that an estimate is an approximation of the value obtained through computation.

2. Recognition that there is a range of values each of which is an appropriate estimate of a computation.

3. Recognition that problem context determines whether or not a particular estimate is appropriate.

II. SKILL LEVEL FACTORS

A. Process Factors

1. Reformulation
    a. Rounding
    b. Truncation
    c. Averaging
    d. Changing form of number, e.g., fraction to decimal

2. Compensation

3. Translation

B. Product Factors

1. Determination of the correct order of magnitude of the product of a computation.

2. Determination of the range of acceptable estimates of a computation.

III. RELATED SKILLS

A. Working with powers of ten.
B. Recognizing place value.
C. Comparing numbers.
D. Mental computation.
E. Basic facts.
F. Properties of operations.
G. Recognizing effect on computation of modification of numbers.

VI. ATTITUDES

A. Confidence in ability to do math.
B. Confidence in ability to estimate.
C. Tolerance for error.
D. Estimating seen as useful.

Figure 1. Model of Hypothesized Developmental Factors

in Parts I and II of Figure 1. Conceptual factors A1 and B2 involve recognition of the existence of more than one appropriate process for obtaining an estimate and that more than one estimate is appropriate. To test these factors, students were shown illustrations where two individuals rounded addends in different ways and/or obtained different estimates. In each situation, the rounded addends and the estimated sums would be acceptable to individuals satisfying the parameters of the Reys et al definition. Depending on the context, students were asked whether the rounding processes were acceptable (A1) or whether either sum was "about right" (B2). The majority of students at each of the four grade levels found both processes acceptable. A majority of the 12 students at each grade level did not find both sums acceptable: 1 student in grade 3 found both sums acceptable; 3, grade 5; 5, grade 7; 6, grade 9. In grades 5 and 7 students finding the processes unacceptable explained their position by referring to the rounding procedure taught in their school and/or to whether the resulting numbers were multiples of ten. In grade 9, students preferring one sum over the other selected the sum involving a fraction (e.g. 10 1/2) because fractions are closer and therefore "better". These data suggest that children are more amenable to alternate processes in estimation than they are to alternate products and that acceptance of alternate products is slower to develop.

Conceptual factors A2 and B1 involve recognition that computational estimation processes involved computing with approximate numbers (A2) and that the product of an estimation is in itself an estimation (B1). To test these factors, situations involving hypothetical students S and V were used: Student S rounded first, then computed an estimate; Student V computed first, then rounded to obtain the same estimate. Interviewees were asked to compare, contrast and establish a personal preference for the hypothetical situations involving students S and V. At all grade levels, interviewees found it acceptable to use rounded numbers as addends or factors (A2) and agreed that estimated sums or products were acceptable (B1). However, a majority of students at the three upper grades preferred to compute with exact numbers and to round the result to obtain an estimate. These students (5 in grade 5; 7, grade 7; 8, grade 9) claimed such procedures were "more accurate" or "easier because you only have to estimate once". It appears that by grade nine, young adolescents do not recognize that an essential component of computational estimation involves computing with approximate numbers or, at least, do not feel comfortable with this procedure.

Skill level factors (Part II of Figure 1) were also investigated through the use of D-problem and O-problem types. Only the former are discussed here. When presented an addition problem where in one case the addends were truncated and in the other case where addends were rounded to the closest hundred, the youngest students found both procedures acceptable. Students in grades 5 and 7 resisted the truncation methods in preference to rounding methods. Seventh grade students when shown a procedure which compensated for truncation errors, agreed that truncation with compensation was as good as, perhaps preferable to, rounding methods. This pattern held with ninth grade students too. In other situations, with other items, half of the third grade students and at least ten of twelve students at the higher grade levels recognized that compensation led to closer estimates. An item given to seventh and ninth grade described two cases involving decimal multiplication: In one case, a hypothetical student incorrectly used a calculator to determine the product; in the other, the decimals factors were changed to approximate fractions. When asked whether the results were acceptable estimates, interviewees split between the two alternatives and lacked confidence in their responses.

Some items testing skill level factors involved estimates with respect to order of magnitude and to a range of acceptability. Results show the ability to relate order of magnitude to the computed or estimated result is a developmental ability perhaps dependent upon an ability to compute mentally. Generally items requiring identification of a range of acceptable estimates were unsuccessful. Interviewees tended to do the calculations mentally before responding to questions concerning whether or not the computed answer was above or below a target number.

Implications. The results of the analysis indicate clear developmental trends on several of the factors identified in the model in figure 1, and lend support to Cases's hypothesis that pre-adolescent children have not developed the prerequisite skills and concepts necessary to be good estimators. This observation should not be interpreted as saying that the recommendations for increased emphasis on computtational estimation are inappropriate. Rather, an implication is that instruction should be directed toward assisting children develop the prerequisite lower-level concepts and skills which form a foundation for later learning of computational estimation as both a process and as a product. To expect children to become proficient at computational estimation prior to the establishment of such a foundation is to invite failure and frustration.

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The Value of the Computer in Learning Algebra Concepts

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The Background

The problems and difficulties which many secondary schoolchildren have with algebra (generalised arithmetic) are well known and have been the subject of much investigation. Many of these relate to the conventions of the notation and the inability of children to interpret the meaning of the use of letters (Booth 1983a). Faced with a new and daunting cognitive situation, many fall back on their previous experience and make use of a one-to-one correspondence between the natural numbers and the letters of the alphabet (eg Wagner 1977), feeling a strong need for a numerical 'answer'. Booth (1983b) reported encouraging success using an imaginary "Maths Machine" which the children had to 'program' to produce answers. The value of computer programming in understanding algebra has already been shown (see, for example, Tall 1983) and the natural extension of Booth's work was to provide the children with actual "maths machines" to program.

The psychological framework of the research is based on constructivist Piagetian theory, with its idea of abstraction from experience, Ausubel's theory of meaningful learning, and the relational understanding of Skemp. All these theories emphasise the importance of the ' framework of knowledge' which the individual constructs in any cognitive area, and the need to build on the existing knowledge structures of the child by conceptual rather than rote means.

The Experiment

a) Equipment

To enable the children to construct a mental model for a variable in algebra, and the manner in which it is manipulated, a concrete model was provided consisting of a 'box' containing the current numerical value of the variable and an attached label with the variable name (figure 1). Although this model does not fulfil all the mathematical uses of the concept of variable (see e.g. Wagner 1981), it proved to be of great value to the children.

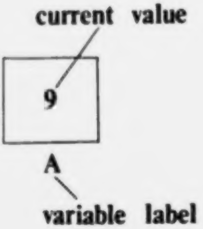
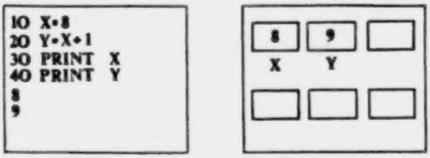


Figure 1

Two "Maths Machines" were devised. The first was a cardboard model consisting of two large sheets of card (figure 2), one of which was blank (the 'screen') and the other with six rectangular boxes to store the variables. To carry out commands placed on the screen, the children performed individual tasks such as carrying messages, looking after variable labels, inserting values on cards into the stores and performing the arithmetic calculations. (See Thomas 1985 for further details.)



```
10 X=8
20 Y=X+1
30 PRINT  X
40 PRINT  Y
8
9
```

The Screen

The Variables

The 'Maths. Machine'

Figure 2

A second, problem-solving tool, designed specifically for the programme was a software 'Maths Machine' for use on the BBC computer. This program, which allows normal algebraic input (with implicit multiplication), was also based on the fundamental mental model of a variable discussed above. The screen consists of a series of 'boxes', initially empty. Some are labelled with variable names and contain the current value of the variable under consideration, others are for algebraic expressions which can be calculated and compared (figure 3). The two 'Maths Machines' were designed to enable the children to develop their understanding of the general concepts of algebra, through structured exploration of practical examples. Each is a 'generic organiser' in the sense of Tall (1985). Through practical experience, specific examples are seen to be generic examples (representatives of a class of examples) from which the general concept may be abstracted. Both turned out to be extremely popular and successful.

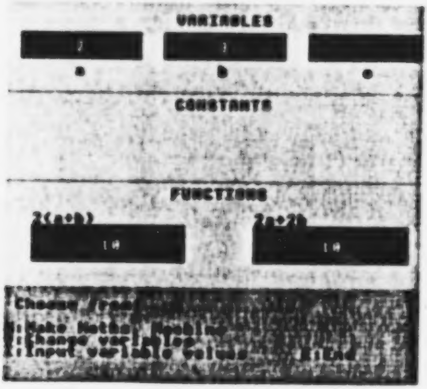

VARIABLES
2
3
a
b
c
CONSTANTS
FUNCTIONS
2(a+b)
10
2a+2b
10
Choose from
Make Maths Machine
Change variables
Input variable values
End

Figure 3

b) The Experimental Method

The subjects of the main experiment were a group of 42 mixed ability 12 year olds from the top year of a middle school, with no previous experience of algebra. They were divided into matched pairs using the results of an algebra pre-test based on the Concepts in Secondary Maths and Science (CSMS) algebra test. The teaching programme given to the experimental group consisted of about twelve hours of work replacing their normal mathematics periods. The module of work consisted of a variety of activities, using the equipment described above. The children were divided into groups of about three and were rotated each session between the computers available (three) and the 'Maths Machines'. The use of small groups was found to have beneficial effects. Peer group interaction, in helping and correcting each other, certainly seemed a valuable means of intelligent learning (Skemp 1985). The pupils started with an introduction to simple programming in BASIC and this was built into some investigations using short programs. An example of the sort of thing looked at would be a comparison of the outputs of these programs for three different values of each of a and b :

```
10 INPUT a               10 INPUT a
20 INPUT b               10 INPUT b
30 c = 2*(a + b)         30 c=2*a+2*b
40 PRINT c               40 PRINT c
50 GOTO 10               50 GOTO 10
```

In this way concepts such as commutativity, the use of brackets and equivalence of expressions were all investigated unobtrusively and linked to practical experience through everyday problems.

The final part of the programme of activity involved the use of the software "Maths Machine" to find the 'solution' to relatively difficult inequalities such as :

For what value or values of x is $2x+1>5$?

This was achieved by inputting the formula $2x+1$ as a function and then choosing values of x to input. The 'Machine' displayed the value of the function for this value of x and so values giving a result greater than 5 could be recorded. It was not expected that many would obtain a result such as $x>2$ from their lists, although some did.

The set of five worksheets used in the programme will be made available at PME 10.

A test based on the CSMS Algebra test, but different from the pre-test, was given as both post-test and delayed post-test ten weeks later.

The Results

The main question under test was whether or not the teaching programme had improved the children's understanding of the use of letters in algebra, with particular reference to their use as generalised numbers and variables.
The results (Table 1) showed that both the post-test and delayed post-test results of the experimental group were significantly better than those of the control group.

TEST (max.=71)	EXPERIMENTAL MEAN	CONTROL MEAN	MEAN DIFF.	S.D.	N	t	df	p
POST-TEST	32.55	19.98	12.57	10.61	21	5.30	20	<0.0005
DELAYED POST-TEST	34.70	25.73	8.47	11.01	20	3.13	19	<0.005

Table 1

The questions involved an understanding of all four of the levels of difficulty identified by Kuchemann (1981). The experimental group were significantly better than the control group on questions requiring an understanding of the use of letters as a specific unknown and as a generalised number or variable (Kuchemann's levels 3 and 4). It was also encouraging to see that in some areas, where comparison was possible, the experimental group results were comparable with or better than those of children up to three years older on the published CSMS results. There were also many very encouraging examples of great individual improvements in understanding of the use of letters in algebra.

The children enthused over all the work, and were still talking about it a year later. The teacher who taught it was equally enthusiastic commenting that it "was a very worthwhile project which proved to be very pupil orientated. It is enjoyable, interesting and thought/discussion provoking between pupils and between pupils and the teacher."

It was concluded that the programme had been successful in its aim and that work of this sort using the computer and presented to secondary school children before they do any formal algebra could have wide ranging benefits.

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INTRODUCING ALGEBRA TO "LOW-LEVEL" 8TH AND 9THGRADERS

Gérard VERGNAUD and Anibal CORTES, C.N.R.S.*

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There has been many studies on algebra during the last 10 years. Not many of them include experimentation and observation in the class-room, and practically none of them is concerned with introducing algebra to "low-level" students ; by "low-level" we dont't mean that these students are not intelligent, only that they are weak. The problem of raising the level in mathematics of the whole population of young people is a problem for all developed countries to-day. It is a difficult challenge.

Not only is it interesting to teach algebra to low-level students for political, economical and social reasons, but also for scientific reasons : the sort of difficulties met by this population of students is somehow an amplified version of the difficulties met by other students. What we try to do in this paper is to approach the problems raised by the meaning, the function and the concepts of elementary algebra. These problems are psychological in nature, as they have to do with the cognitive and motivational aspects of learning algebra. They are also epistemological, as a crucial point in this business is to identify to what kind of problems algebra brings an answer that can be made meaningful to students when they are first introduced to algebra.

Algebra is a real epistemological shift from arithmetics : instead of handling a natural language problem with intuitive tools (theorems-in-action), students have to manipulate chains of symbols with explicit rules. So there are several dimensions of the shift :

- explicit/implicit

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*This research will be published more extensively in "Recherches en Didactique des Mathematiques". Pierre Favre-Artigue, Pierrette Serrano, Raymond Piquer, Jean Genais and Ying He collaborated actively in this research.

- symbolic language/natural language
- algorithmic/heuristic

Moreover the shift cannot be made without the introduction and the use of powerful concepts such as : unknown and equation, variable and function, directed, rational and real numbers, abscissa, coordinate and graph, monomial and polynomial, system of equations...

When one introduces algebra as a formal part of mathematics, which has to be learned anyhow, there is undoubtedly a shift from what students have done before. But then, the commun function of algebra and arithmetics, which is the solving of problems, is hidden.

As we have made the choice, in this experiment, to introduce algebra as a way to solve arithmetic problems that would not be easily solvable without algebra, the shift is different : algebra has to do with the same problems as arithmetic, but uses different procedures and conceptual tools : it is a big shift from arithmetic to put a situation into equations, to use algorithmic procedures to transform and combine these equations in order to find a solution if there is any, or to show that there is no solution ; in the usual arithmetic procedures one chooses a sequence of operations (sometimes one by one) to calculate intermediary unknowns, until one is able to calculate the last one. In an arithmetic solution, intermediary unknowns must be meaningful ; whereas in an algebraic solution, once the relationships of unknowns with data have been expressed and written down (more or less as relations between variables), one does not have to care about the meaning of intermediary expressions until one gets to the solution.

In that sense, algebra is a "detour" : students must give up the temptation to calculate the unknown as quickly as possible, they must accept to operate symbols without paying attention to the meaning of these operations in the context refered to. They must also understand the equality sign differently from what it meant before to them, they must operate with brackets and do several other new things like add and subtract, multiply and divide on both sides, so as to isolate the unknown on one side and the numerical data on the other, or to eliminate one unknown.

CONCEPTUAL PROBLEMS RAISED BY ALGEBRA

Different coneptual problems are raised by algebra, that are echoed in students' cognitive difficulties.

1 - The meaning of the equality sign : does it announce a result, or represent a numerical identity, or an equivalence between two combinations of numbers, or an identity of functions...

2 - The autonomy of symbols and symbolic operations : the use and manipulation of symbols is essential in algebra. It is very rare in natural language that students manipulate words without considering their meaning. In algebra, it is both necessary and dangerous : examples will be given.

3 - The powerful and difficult concepts of variable and function : the concepts of unknown and equation are not self-sufficient ; there are algebraic operations that cannot really be conceptualized without some explanation on variables and functions. The example of formulas in geometry, physics, or elementary economics, is a good way to show some distinctions.

4 - The meaning of negative solutions

How is it possible to get students solve (linear) equations in the set of real numbers, if they cannot give any meaning to negative solutions. In order to make negative solutions meaningful when expressing algebraically concrete situations, unknowns must represent not only magnitudes and quantities (they are always positive) but also relationships and transformations (they can be positive or negative).

5 - The manipulation of letters and the notion of system

Several authors have shown the difficulty raised by the presence of a letter inside expressions (lack of closure : Collis), on both sides of the equation (Filloy and Rojano, 1984 and 1985). This usually hides the fact that there is not only one unknown but two or more. It is also in those cases that algebra is a powerful tool, comparatively with arithmetic. So it may be valuable to introduce rapidly systems of equations, so as to show the power of algebra, and its difficulty...

A DIDACTIC EXPERIMENT

We worked for two years with a class of 21 students in a vocational school. The experiment consisted of two series of lessons that took place during the 8th grade (12 hours in all) and during the 9th grade (10 hours in all). There were a pre test, an intermediary test and a post test, but the most interesting data have been collected in the class-room, by observers with audiotapes : there was one observer for three students working either individually or together. The observer was also a participant and helped the students when necessary.

The didactic situations were chosen so as to face the students with the alvove-mentionned (1 to 5) conceptual problems.

First year : 8th grade

- The equilibrium between weights was used as a physical model for some situations

from $a + x = b$ (all positive)
to $ax + b = cx + d$ (all positive)
and even $a(bx + c) + d = a'(b'x + c') + d'$ (all positive)

- The model of additive and subtractive transformations and the model of change in temperature were used to introduce negative solutions, and negative data.
- The "unequal sharing" paradigm was used to introduce simple systems like the following :

$$\left| \begin{array}{l} x + y = a \\ x = y + a \end{array} \right. \qquad \text{or} \qquad \left| \begin{array}{l} x = a\,y \\ x = z + b \\ z = y - c \end{array} \right.$$

- The diagrams representing the program of calculation (on a pocket Hewlett-Packard) of the equations obtained were also used.

Second year : 9th grade

- Functions of two variables were systematically used to introduce two-unknowns-and-two-equations systems, and Hewlett-Packard calculation programs.
- Some work was done on formal expressions

$$\left| \begin{array}{l} f(x, y) = ax + by \\ f(x, c) = d \end{array} \right. \qquad \text{find } x$$

or $\left| \begin{array}{l} f(x, y) = ax + by \\ f(x, y) = c \end{array} \right.$ find different solutions for x and y

- The graphic meaning of such solutions was introduced
- Natural language problems were then used again, corresponding to the above-mentionned formal expressions, before the classical problem

$$\left| \begin{array}{l} ax + by = c \\ a'x + b'y = c' \end{array} \right.$$

was also introduced through a natural language problem with its graphic representation, and the algorithmic solution (linear combination).

- Finally five natural language problems were presented, the arithmetic solution of which would not likely be in the reach of most students.

The protocols that we collected are rich and varied. They illustrate the points raised in the first part of this paper, and we observed many difficulties that other authors have mentionned before ; we also find new facts concerning the "detour" through algebra and "the didactic contract" by which students progressively understand what is required from them by the teacher.

Important objects to be considered are the "scripts" that students have to learn and master. By "script" we mean a scriptural scheme which is the symbolic expression of the solving procedure. Conceptual difficulties may lie either in the validity of the script or in the progression to the goal. These difficulties are distinct in nature. Examples will be given.

Finally it is interesting to notice that the last problem in the sequence was solved by all students. But the tests show that the overcoming of difficulties in algebra, especially when negative numbers are involved, is a long term learning process.

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5. THE MATHEMATICAL LEARNING ENVIRONMENT

MATHEMATICS BACKGROUND TO TECHNOLOGY : THE CASE OF NIGERIA

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INTRODUCTION

Nigeria recognised the importance of technology in her development a long time ago. The National Policy on Education (1981) provides the operational framework for her education. The importance of mathematics in technological development is reflected in the statements of the purpose and objectives of education.

The National Science and Technology Development Agency in Nigeria has estimated that by 2000AD the country's population may well exceed 140 million people hence proposed a systematic approach to science and technology for a self-reliant and endogenous development (NSTDA, 1979). It had since suggested the intensification of education and training relevant to Nigeria's gradual build up of her own internal scientific and technogical capacities since technological transfer is a myth. For example, the NSTDA (1979) indicated that the importation of ready-made technology stifles national initiative and leads to a heavy commitment of Nigeria's foreign exchange. The mathematics background required for developing a viable indigenous technology is enormous. Technological problems are indeed real life problems which rarely match the rigid boundaries, traditionally imposed on mathematics.

MATHEMATICS TEACHING IN NIGERIAN SCHOOLS

Nigerian educators are concerned about students' antipathy to mathematics. Lassa (1984) devoted his inaugural lecture to 'the sorry state of mathematics education in Nigeria'. A particular section of this lecture succinctly presents the current state of mathematics in Nigerian schools as "The performance of students in mathematics has been declining, the students' attitude towards mathematics tends to be negative, most of the teaching staff in mathematics have been inadequate and ill prepared for the teaching of mathematics and the society had the feeling that mathematics is for those with strange things up in their heads. All these have left mathematics teaching and learning in a deplorable state of affairs in all institutions of learning." (Lassa, 1984, p.1)

There is also abundant evidence on the declining trend in students' performances in mathematics in the West African School Certificate Examinations (Fakuade, 1973; New Nigeria, 1978; Uzoma, 1980; and Lassa, 1984). Poor performances in mathematics have far reaching effects on other school subjects of which the sciences are very important. Technology, briefly described means the systematic application of scientific and other organised knowledge to practical tasks.

The impact of mathematics on science is great and when students do not develop positive attitudes to mathematics, they find it impossible to cope with the sciences which invariably are the base for technology. The poor attitudes of pupils may be linked with other factors like;

a. Mathophobia in the classrooms
b. Pygmalion in the classrooms and mathematics teaching
c. Teacher qualification
d. Dearth of instructional resources
e. Context of mathematical presentations
f. Some myths in Nigeria about mathematics

a. Mathophobia in the Classroom

Literature is replete with information on phobia which means fear (Freud, 1909; Kristel 1981; Sluckin 1979; and Lewis & Rosenblum, 1974). Mathematics is a school subject which is feared by many Nigerian students and in which they take avoiding actions. The older generations of Nigerians learnt mathematics under duress (in some cases). Teachers in those days held whips in front of the class during mathematics classes thus the learning of mathematics was more of an ordeal than a pleasant learning experience for the child. Even today in some schools, the practice still continues.

Bowlby (1973) indicated that the cause of fear may be linked with the presence or absence of something that provides safety and security. A teacher with a whip in his/her hand in front of a class does not provide safety or security. The presence therefore of a 'whip-holding mathematics teacher' often raises the state of anxiety of many Nigerian children. The propensity for damage of this kind of classroom set-up is incalculable. In Akinyemi (1980) many pupils indicated that their fear of mathematics was linked with teachers' threatening approach to the teaching of this subject.

b. Pygmalion in the Classroom and Mathematics Teaching

Pygmalion in the classroom is an important aspect of classroom interaction which had been espoused by Rosenthal and Jacobson (1968). The theory holds true especially in the teaching of mathematics in Nigeria. In a study to investigate classroom interaction (self-fulfilling prophecy) conducted by Akinyemi (1980), 150 class II pupils, distributed in four centres of the University of Ilorin Summer School Programme (SSP) were used. The syllabus represented a review of class I and an introduction to class II mathematics curriculum of the Nigerian secondary schools. All pupils in the four centres took the same pretest and posttest. All mathematics teachers used the special syllabus for the four weeks programme. Three centres (A,B,&C) were informed about their pupils' pretest performances but were not shown. The fourth centre was used as the control. Unobtrusive observations of teacher interactions were conducted during the second and third weeks of the study.

Pretest results reflected very close averages (x = 34%) for all four centres being homogeneous groups. A deliberate bias was introduced on the second day of the study to investigate self-fulfilling prophecy as;

- Teacher in Centre A (Ta) was informed that he had very good pupils in his class and that their average score on pretest was Xa = 70%.
- Teacher in Centre B (Tb) was informed that he had extremely weak pupils in his class and that their average score on pretest was Xb = 25%.
- Teacher in Centre C (Tc) was informed that he had average pupils in his class with an average pretest score of Xc = 34% (being also the average score for the entire group).
- Teacher in Centre D (Td) was not informed about his pupils' performances and did not bother to ask. However, the average pretest score of the group was Xd = 34%.

Posttest results showed a gain of X = 21% over the pretest with average group scores being Xt = 55%, Xa = 57%, Xb = 50%, Xc = 53% and Xd = 56%. The results of this study did not show any statistically significant differences. However, in terms of numerical value, centre A was best (as expected) followed by centre D which was the control. Centre B which was presented as the weakest class in mathematics actually performed the poorest.

Teachers A and D were found to be enthusiastic judging from their presentations, reinforcements methods, class participation and the general classroom atmosphere during the second and third weeks' unobtrusive observations. Teachers B & C did not show any enthusiasm. Their attitudes were at best lukewarm. The feedback received from these two (Tb & Tc) after the study was that "four weeks of mathematics teaching along with other subjects was not sufficient to make a noticeable impact on pupils who are inherently weak". One of the enthusiastic teachers(Ta) confessed that he felt like helping the pupils because he was delighted to learn that pupils in his class were the best of the four centres and wanted them to maintain their superiority on the posttest.

c. Teacher Qualification

A critical look at the way mathematics is taught in the Nigerian schools reveals many shortcomings. In many instances the teachers are either unqualified or underqualified to teach the subject. Adesina (1980) has revealed startling statistics on the state of primary school teachers in Oyo state of Nigeria which shows 34.5 per cent as unqualified and 50.7 per cent as marginally qualified (qualification below Grade II level which is required to teach in the primary school). The situation in many secondary schools is equally bad. One can then imagine the cumulative effects of poor teaching of mathematics (an intellectual problem-solving skill) on the mathematical background and subsequent development of the child. Towards this end, the government has made it mandatory for all teachers to be trained within a short time.

d. Dearth of Instructional Materials

The scarcity of materials and resources in schools had long been expressed in several quarters (Fakuade,1973; Lassa, 1984; West Africa, 1985; and Harris & Akinyemi, 1986). The need to provide materials which are relevant to the Nigerian situation was the basis for the set up of the Textbook Development Agency. A series of curricula changes have occurred in schools since Independence in 1960 and the necessity to produce materials in tune with the new syllabuses had long been felt since the previous materials used (Durell mathematics series) had become inappropriate for the Nigerian local conditions.

e. Contexts of Mathematical Presentations

The concern for presenting learning materials in meaningful contexts is one that psychologists (Ausubel,1968 and Bruner, 1966) had since expressed. Ervynck (1983) said that the learning difficulties of the African student in mathematics may "relate to some unacquaintance with classical features, materials and examples used to describe mathematical concepts". Materials and features available in different learning settings must be used in presenting concepts otherwise learners are left confused either in the abstract presentations or in the use of foreign contexts which are unfamiliar to them.

f. Some Myths About Mathematics

In changing the Nigerian students' poor attitude to mathematics certain misconceptions and myths need to be corrected in schools. It may take some effort on the part of the teacher to change the situation. Lassa (1984) has presented one such myth as "mathematics is for those with strange things in their heads". The statement that "if you work too many mathematical problems, you may go mad" had been expressed by some young learners who had been so informed by their illiterate parents. The myth that "you must be born with mathematics to be good at it" is one that is popular among the Nigerian secondary school students. Counting, especially for young learners is an early skill in mathematics. In situations where you count the number of children in an illiterate family in a rural area, a serious problem may arise because it is a myth in some tribes for anyone to count their number of children! It is obvious that the concept of probability in statistics will suffer in such an environment because the issue of 'chance' is left to the 'gods'!

CONCLUSIONS

Some inadequacies in the educational systems have been indentified and are being corrected. The campaign in schools is towards reviving mathematics and the slogan is "Mathematics is a friend and not a foe" Technology and development go hand in hand and the Nigerian curricula at the secondary school level have recently been modified to promote technology. In the continuing search for solutions to children's antipathy towards mathematics, resources including games and simulation exercises are being designed to stimulate children's interest and to sharpen their computational abilities in the basic mathematical operations. These provide a sound foundation which is sine qua non to the development of a receptive attitude to the subject and its application to science and technology.

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OUTCOMES OF THE DIAGNOSTIC TEACHING PROJECT

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Different aspects of our research on diagnostic teaching have been reported at PME in each of the last years (except 1985). A summary report is now available (Bell, et al, 1986); this talk will offer a brief account of the main outcomes, with more detail of some aspects which have not been discussed previously. The aim has been to develop a way of teaching which contributes clearly to long term learning and which promotes transfer. The key aspects of this method are the identification and exposure of pupils' misconceptions and their resolution through 'conflict-discussion'. Conceptual diagnostic tests also play a part both in helping pupils to become aware of their misconceptions and enabling the teacher to observe progress.

First we shall report experiments which test the effectiveness of the major features (focus on misconceptions, conflict and discussion) of the diagnostic teaching methodology, in comparative teaching experiments. Secondly, there are smaller scale experiments testing other aspects, such as the use of diagrams, substituting easy numbers, and immediate feedback. Thirdly, there are sequences of experiments, on each of the three main curriculum topics considered, through which the overall conceptual diagnosis, the design of the key types of task and their mode of use have been tested and developed, and the general effectiveness of the teaching units has been improved. Fourthly, there are experiments in the application of the full methodology to further curriculum topics (algebra, probability, ratio, measure conversion, the simplest addition and subtraction problems, shape recognition). The outcomes of these experiments have been developments in the methodology of teaching design, knowledge about the value and feasibility of different types of pupil task, (such as Making Up Questions, Marking Homework, Group Working), and also new information about pupils' concepts and common misconceptions in these topics. Fifthly, two larger scale tests of understanding have been conducted, covering (a) Multiplicative Problems and (b) Directed Quantities and Numbers, to indicate how widespread in the population are the misconceptions we identified in our experiments with smaller numbers of pupils. Finally, there has been work aimed at the wider dissemination of the teaching methodology; packages of materials with teachers' notes have been tried with teachers having little or no previous experience of the methodology, and improved versions have been produced. Videotapes of pupils interviews and of teaching episodes have been made; and a group of teachers has been established to develop the application of the methodology to further curriculum topics.

TESTS OF THE DIAGNOSTIC TEACHING METHODOLOGY

1. Conflict vs. Positive-only (Decimals)
2. Amount and Intensity of Discussion (Directed Quantities)
3. Diagnostic vs. Expository (Rates)

The research programme has included three comparative experiments demonstrating the greater effectiveness of a teaching sequence containing one or more features of the diagnostic method. The first of these showed the superiority of a 'conflict' as against a 'positive-only' approach to the teaching of decimal place value; the positive-only approach focussed onthe areas which were known to cause difficulty, and correct concepts and procedures were evolved without explicit discussion of misconceptions, while the conflict method first led the pupils into exposing their misconceptions before holding a discussion leading to their resolution. The second experiment showed that, of seven classes using similar teaching material but with varying degrees of conflict discussion, the more vigorous and intensive discussions were associated with greater progress. The third experiment showed greater learning in seven diagnostically taught classes compared with two taught by 'exposition for understanding'.

TESTS OF PARTICULAR ASPECTS OF THE TEACHING METHOD

1. Using Diagrams
2. Substituting Easy Numbers
3. Games
4. Making Up Questions
5. Marking Homework
6. Group Tasks
7. Immediate Feedback

1. USING DIAGRAMS

Drawing a diagram is a standard way of trying to solve a problems and is often used to help to clarify problems for pupils. Our experiments have shown that the use of diagrams is not as straightforward as is generally assumed. Broadly speaking, those pupils who need a diagram to help them solve the problem are those who cannot draw a correct one, because they cannot conceptualise the problem sufficiently clearly. Conversely, those who could draw a correct diagram often do not need to, because they have the necessary understanding to solve the problem mentally. Thus the notion of the diagram as a solution method breaks down. However, teaching which used the construction of a suitable diagram as the basis of a discussion of the concepts concerned and their relationships did appear successful. This clarified the understanding of the situation and pupils were then often able to solve the problem without needing to draw any diagram. This provides some confirmation that it is the explicit discussion of the key concepts which is essential to learning.

2. SUBSTITUTING EASY NUMBERS

We have studied the substitution of numbers such as 3, 6 in problems containing numbers such as 28.7, 0.4. In easy number problems the choice of operation itself is easier, apparently because of the possibility of rapid trials and checks of consistency with expectation. We also know that the pupils do not necessarily regard the operation as invariant under changes of number in a problem. Pupils regard the operation as residing in the numbers rather than in the problem structure. For example, $8 \div \frac{1}{4}$ may be seen as essentially the multiplication, 8×4, which is the calcuation which actually needs to be performed. Difficulties also arise in making suitable choices of 'easy number': 0 and 1, for example, are unhelpful. Our work has shown that it is preferable not to regard substituting easy numbers as a solution algorithm, but to vary the numbers in a problem as a means of developing the concept of the invariance of the quantity relations under changes of number.

3. GAMES

Games which engage the players in choices involving the key concepts and misconceptions have been developed and are clearly very powerful learning situations. If well designed, they have the elements of checking (either inbuilt or by opponent's challenge), self-adjustment to a pupil's own level, one is essentially choosing one's own examples, and of repetition with variety. Successful games were developed in most of the teaching experiments. To achieve the potential value of a game it is necessary to follow it by a discussion focussing on the principle to be learned or the misconception to be overcome, and in which it is explicitly articulated, by pupils as well as by the teacher.

4. MAKING UP QUESTIONS

Making Up Questions includes several ways of reversing the usual teaching order of the teacher asking the questions and the pupil providing the answer. One type is the generation of questions by the pupils of an initial situation; another is the giving of a calculation say '0.4 ÷ 25' for which the pupils have to make up a 'story' possibly in a given context, such as speed.

5. MARKING HOMEWORK

Marking Homework is another type of reversal of role where the pupils mark another's actual or fictitious script, stating the nature of the error and offering a possible explanation.

Both Making Up Questions and Marking Homework have proved good ways of provoking reflection and discussion, though they are not easy tasks, and need some perseverence for the pupils to get used to them. Examples of the development of more successful forms of these tasks will be found in the reports of the of the later teaching units.

6. GROUP TASKS

A development of class discussion which has been found successful with many teachers and classes is to set the critical problems first to be tackled by small groups of pupils. After they have arrived at group conclusions, the class is brought together and each group explains its conclusion to the class, using the blackboard as appropriate. This can encourage greater willingness on the part of pupils to express views about which they are not entirely sure, while retaining the opportunity for the teacher to be aware of wrong conclusions and to challenge them. One particularly useful type of group task is the completion of a table by placing small cards in the appropriate cells. Groups discussion focuses naturally the correctness of placing and reconsideration is easy.

7. IMMEDIATE FEEDBACK

Gelman (1969) obtained a striking level of success in teaching number and length conservation by simple yes/no feedback of correctness. We found that this did produce improvements, even after relatively short experience. Boxes were provided for recording the chosen operation and calculator answer, with additional boxes for further attempts if the first one proved incorrect. Test results showed modest improvements, particularly on certain test items. Pupils did appear to use the feedback intelligently, by reconsidering the question, but learning was limited because they lacked means of obtaining a more correct view of the problem. In conjuction with feedback, some more positive teaching input is required.

NUMBERS AND NOTATION VERSUS PROBLEMS IN CONTEXT

This sequence of experiments pursues a research question. The initial question arose from the observation that choice of operation in problems was heavily influenced by numerical misconcpetions. The question was whether teaching aimed at removoing these would be more or less effective than teaching whose purpose was to establish the general quantity relations in context, e.g. weight x price = cost. The first experiment focused directly on this question comparing two classes, one having each of these treatments. Neither class made very much progress on a choice of operation test. The second experiment used improved teaching material combining the two approaches. This produced impvoements on price questions (the most emphasised context in the teaching) but little else. The third experiment used a much stronger unit of teaching material (Numbers and Notation) (NN) aimed at the numerical misconceptions only. This produced quite substantial gains on the numerical section of the test, but no change on choice of operation. The fourth experiment used a new unit entitled Problems in Context (PC) aimed at dealing with the numerical misconceptions within the context of problems concerning price, and the fifth used Numbers and Notation followed by Problems in Context. The results of these showed, in the first case, good gains on choice of operation as well as on numerical questions from the use of PC only, and in the second case, trivial gains on choice of operation and substantial ones on numerical questions at the end of the teaching using the NN unit, with further substantial gains on both types of question following the PC teaching. It thus appears that treating the misconcpetions in the desired contexts is essential, and that this effect transfers to the numerical questions but not vice versa.

In the talk, there will be an opportunity for questions to be raised, and for further details of these experiments to be given, as required.

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A Study of the Socialization to Teaching of a Beginning Secondary Mathematics Teacher

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This study was designed to determine the crucial elements involved in the socialization to teaching of one beginning mathematics teacher and thus to deepen our understanding of the socialization process. Teaching is a complex activity involving not only the teacher but also other persons with whom he or she interacts. Although the process of socialization to teaching is unique to an individual and his or her teaching situation, there are certainly elements involved in this process that are common to the experiences of many teachers. This report will focus on one element that appeared to be crucial to the teacher participant's socialization experience - his perceptions of the students he taught.

The specific questions that guided the study related to the influences of the following factors on the actions and thoughts of the teacher in the classroom: conceptions of mathematics teaching held by students, parents, administrators, and other teachers; course content; the teacher's perceptions of students' maturity and ability levels; the teacher's biography, including his teacher education program and student teaching; and the teacher's plans for the future.

Data collection began while the teacher was a preservice master's degree student in mathematics education and continued to the end of his first year of teaching. The teacher was interviewed extensively during his final year of teacher training at the university, I wrote a case study and the teacher, Fred, reacted to the case study. Extensive interviewing and participant observation were used as primary data sources during the eight weeks I spend with Fred during his first year of teaching. Supplemented with questionnaire data and artifacts such as lesson plans, tests, and school and community newspapers, the interviews and observations provided a data base from which I could understand the teacher's classroom actions and how he and his students thought about them during his first year of teaching.

Given the variety and the amount of data collected during the study, and my desire to study those aspects of the situations that seemed to be most critical or significant to the participants, some analysis had to occur while the data were being collected. Field notes and interview transcripts were analyzed for emerging patterns of activities and ideas soon after they were generated. This information was then used to guide future data collection. Coding categories were developed in order to help organize the data, and hypotheses were formed and tested concerning the significant forces in Fred's socialization process.

The data suggest that the way the teacher thought about his role in the classroom and performed that role during his first year of teaching was most significantly influenced by his university teacher education program and by his students, that is his perceptions of his students' conceptions of mathematics teaching and their maturity and ability levels. A conception of mathematics teaching may be thought of as consisting of three components: a conception of mathematics, beliefs about appropriate goals and tasks for the mathematics classroom, and beliefs about the relative responsibilities of teacher and students concerning motivation, discipline, and evaluations. The conception of mathematics teaching developed during his time at the university and his student teaching contributed to the perspective from which Fred initially defined the situation in his classroom. Mathematics and mathematics education courses had provided him with reasons to express the belief that problem solving was the essence of mathematics and heuristics were central to problem solving. He seemed convinced that problem solving would be the means by which he could motivate students to learn mathematics.

Students' conceptions of mathematics teaching interacted with the teacher's conception to influence his teaching actions and thoughts. Students in general seemed to believe mathematics consisted of rules and definitions to be memorized and used to solve assigned exercises. They believed that the teacher should define terms and explain procedures carefully, working examples to show students how the exercises should be solved. Students viewed assignments as a means of practicing procedures and as a means of indicating what they had learned. Some students enjoyed solving the recreational problems that the teacher posed occasionally in class but saw little connection between those problems and the learning of mathematics. Fred believed that his students had a very instrumental understanding of mathematics and very little knowledge of fundamental mathematical concepts and skills.

Thus, the conception of mathematics teaching he believed was held by the majority of his students was in conflict with the teacher's own conception. Early in the school year, Fred used problems, particularly recreational mathematics problems, as a means of motivating his students and presenting some of the mathematics they were to learn, but perceived the responses of students to be negative. It seemed to him that students were attentive only when he was discussing how something was to be done for an assignment that was to be graded and when that something fit their conception of school mathematics.

Fred was able to give evidence to support his perceptions of the students' conceptions. For example, in his general mathematics class he introduced an activity designed to show the relationship of probability and mortality tables using dice. This was perceived by the students as a game, indicating a lack of seriousness on the part of the teacher. Fred cited a statement made by a student in the class as evidence of this attitude: "Mr. Lincoln, we're trying to learn some mathematics here. Why are we playing this game when so many of us are failing? You're supposed to be teaching us."

Even his better students often disappointed him. Although these students seemed to be motivated to learn mathematics, their conception of mathematics seemed very limited. For example, Fred presented a "proof" that 2 = 1 in his senior class. The students watched quite attentively as he worked through the proof, many copiously taking notes. When, after reaching the conclusion that two did indeed equal one, Fred asked the class to find the flaw in the proof, a student responded, "Is there something wrong with it?" Even more surprising was the lack of reaction from the class to this comment. Students did not seem to be disturbed by a "proof" of something contrary to what they believed to be true.

The fact that the conception of mathematics teaching held by most of his students was conflict with Fred's own conceptions had an important effect on Fred in the classroom. Early in the school year he began to perceive that his students were not, and, he believed, perhaps could not be, interested in the same mathematics he found interesting. In an interview he complained:

> Even with the caliber of students in the senior class, there are none in there who are that inquisitive to ask "Why does this work?" or "How do I know this is true?" So I don't push and try to justify everything.

Initially, he was surprised that his students were not interested in the way he presented mathematics. He had been told in education classes that problems could be motivating in the classroom, and he himself enjoyed problem solving, believing it to be the essence of mathematics. As the school year progressed, he developed explanations for his students' lack of interest. He felt that students found it "easier to open a book and learn a few sets of rules and procedures and have the teacher explain to them those procedures that they can then use." It seemed to him that students were not accustomed to mathematics lessons that included developmental or problem-solving segments, that teenagers could not be expected to appreciate mathematics, and that they just were not that interested in the subject matter.

Rather than seek ways in which he could change the students' conceptions of mathematics teaching, or at least encourage them to be open to new possibilities for mathematics , Fred seemed not to make an effort beyond that of occasionally presenting what he considered to be interesting problems in class. His lack of enthusiasm may be rooted in an attitude which the following statement expresses - although Fred was an avid mathematical problem solver, he had very little interest in solving pedagogical problems.

> In teaching you can't say "Well, I've solved a problem." Maybe you can say that, but it's not the same type of problem [as in mathematics]. I mean, you have a problem: How can I reach this students I haven't been able to before? I can do all these exciting things and maybe that's creative teaching, but that's not the kind of creativity that I enjoy. Maybe it's because I enjoy recreational math problems so much; at the end it feels so good that I solved this problem. Personally, I don't get that same charge out of coming up with a dynamite less plan.

As the school year progressed, Fred used fewer problems of any kind in class; tended to minimize the developmental portions of his lessons, concentrating on explanations of procedures; and gave assignments and tests requiring only an instrumental understanding of the curriculum. He expressed the belief that it was only in this way his students would cooperate with him and at least learn the material he presented in this way.

There is some indication, then, that Fred's perceptions of students contributed to a modification of his conception of mathematics teaching, although the extent and nature of the modification is unclear. Even at the end of his first year of teaching, he continued to express his enjoyment of problem-solving activities and the belief that problem solving was the essence of mathematics. That is, his conception of mathematics did not seem to have changed. Perhaps other elements of his conception of mathematics teaching also had not changed considerably. However, by the end of his first year of teaching, his classroom actions were not consistent with the conception of mathematics teaching he had expressed before the beginning of the year.

Learning Environment Differences in the Mathematics Classroom

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The collective approach to formal instruction, in which children are taught in groups of 15-40, has led to an increasing interest in classroom dynamics and the resulting learning environment. Despite the amorphous nature of the concept "learning environment", many attempts have been made to identify measurable properties related to the classroom climate. High-inference measures of this construct have tended to take the form of a multi-facet questionnaire on which pupils rate what happens in their class (Chavez, 1984).

Studies concerning the classroom learning climate have generally been of two kinds: 1) those which seek to establish the determinants of the environment and 2) those which focus upon its impact on both cognitive and affective pupil outcomes. In spite of numerous studies in these areas, much remains to be learned about the classroom environment. More importantly, it is still unclear how educators can influence, or manipulate, the environment to the pupils' best advantage.

The focus of the present paper is the mathematics classroom. Some information already exists regarding student perceptions of learning environments in mathematics classes. For instance, compared to classes in other school subjects, mathematics classes have been found to be more difficult, more cohesive, less formal, and quicker-paced (Anderson, 1973; Welch, 1979). Aspects of the environment which have been found to correlate with pupil learning in mathematics, as well as in most other school subjects, are difficulty, satisfaction, cohesiveness (all positive correlations), speed, friction, and cliqueness (all negative correlations). Competitiveness, although negatively related to learning in the sciences, has been shown to have a positive correlation to mathematics learning (Anderson, 1973; Hofstein & Ben-Zvi, 1980).

In general, research has shown that learning environment variables account for a significant portion (from 13% to 46%) of the variance in pupil achievement in various school subjects (Anderson, 1973; Hofstein & Ben-Zvi, 1980). In the area of mathematics, O'Reilly (1975) found the learning environment to explain 67% of the variance in achievement scores! If classroom climate indeed bears such a strong influence on pupil learning as this suggests, then it is imperative to conduct further research in this area. Not only must the specific factors which affect the climate in the mathematics classroom be carefully identified, but possible ways by which desirable changes can be made must be explored.

The present study was exploratory in nature and was designed to provide greater insight into the learning environment in the junior high school mathematics classroom and a better understanding of factors influencing perceptions of classroom climate. More specifically, this study was carried out in order: 1) to identify the type of classsroom climate characteristic of the junior high school mathematics classroom, 2) to determine differences in mathematics classroom climates when comparing high ability and low ability classes, and when comparing classes studying at different grade levels, 3) to determine whether boys and girls manifest different perceptions of the same mathematics classes, and 4) to determine the impact of an intensive in-service course for mathematics teachers by using the classroom environment measure as an evaluation tool.

METHOD

The Instrument

A mathematics classroom environment measure was developed which examined the following eight properties: competitiveness, goal-direction, formality, speed, difficulty, satisfaction, inquiry orientation, and diversity of instructional materials. The first 6 areas were adapted from the *Learning Environment Inventory* (Anderson, 1973; Chavez, 1984), while the last 2 sub-scales were created specifically because of their relevance to the classroom orientation in many modern mathematics programs.

The original questionnaire contained 39 items, such that each sub-scale was composed of at least 4 items. Items appeared in the form of general statements referring to the whole class: for example, "In my mathematics class, there is strong competition among pupils," "The pupils in my mathematics class feel dissatisfied," or "In my mathematics class, questions are presented for investigation in class." A 4-point Likkert type scale was attached to each statement on which respondents were to rate their class from 1-"it never happens in my class" to 4-"it always happens in my class". Individual sub-scale scores were calculated by averaging the pupil's responses on all relevant items after coding in a unified direction. Class averages were computed from the individual means.

Item-scale analysis was made for the different sub-scales which resulted in the decision to exclude 6 items from further data analyses. Cronbach alpha reliability coefficients for the final sub-scales ranged from 0.47 for both the difficulty and diversity sub-scales to 0.77 for the competitiveness sub-scale.

Procedure

This measure was tried out in 20 classes in 7 junior high schools in Israel (N=402 pupils) in which the Rehovot Mathematics Program was being taught. The teachers of these classes were attending an intensive mathematics in-service course held at the Weizmann Institute of Science during the 1984-85 school year. This course was intended to improve their teaching styles, encourage diversity and inquiry in the classroom, and deepen their own comprehension of mathematics.

Administration of the questionnaire took place soon after the teachers had begun the course. Seven teachers administered the measure to the same classes (N=155 pupils) a second time about 5-6 months later, shortly before completing the course.

RESULTS AND CONCLUSIONS

Numerical and graphical presentations of the results will be shown at the Conference. A summary of findings is given below.

1. The property of the learning environment which was rated highest on the average in all classes was goal-direction, while that rated lowest was diversity of teaching materials.

2. General patterns emerged in which certain properties of the mathematics classroom environment tended to be highly interrelated. First of all, classes characterized by an inquiry orientation tended to be high on goal-direction and satisfaction, and low on speed and difficulty. Secondly, greater diversity of teaching materials tended to exist in classes which were less formal. Thirdly, competitiveness was characteristic of classes which were low on goal-direction.

3. Few differences were found in the learning enviroments of high ability classes as opposed to those in low ability classes. In the former classes, however, pupils tended to perceive instruction as more diverse and learning was seen as more difficult.

4. Learning environments in Grade 7 classes were perceived very differently than in Grade 8 classes. In Grade 7 there was more goal-direction, greater formality, more inquiry-orientation, a slower pace, less difficulty and less diversity of instructional materials. Moreover, Grade 7 pupils perceived a greater general sense of satisfaction in their classes than Grade 8 pupils. It is felt that these differences can be attributed more to curricular differences rather than to pupil-age differences.

5. Boys and girls had different perceptions of their mathematics classes: boys saw their classes as more formal, difficult and competitive than did girls. Since these factors have been shown in previous studies to correlate with learning, it is suggested here that the sex differences in the perception of the classroom environment are probably tied in with sex differences in mathematics achievement.

6. In evaluating the effects of the teacher in-service course on classroom climates, results were rather disappointing. On only two sub-scales (competitiveness and formality) were there changes in the desired direction in the majority of classes (4 out of 7).

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Disparities in the Translation of the Cognitive Tests in the Second International Mathematics Study

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Considerable caution needs to be used when interpreting test results, particularly when the test was administered in more than one language. For valid comparisons among countries to be made, it is not enough to know whether the students have had the opportunity to learn the material on which they are tested and to test all students with the same items. The possibility that the meaning of an item was altered in translation must also be examined.

Two questions must be considered. The first is: Were the students really tested with the same test? In other words, did the translations of the test into the 10 different languages really preserve both the exact content of each item and its language level? This question can be answered by having qualified people check the translations for accuracy and language level.

If, in a translated item, the level of difficulty of the language in which the item is couched is judged to differ from that of the original, then a second question must be asked: Did the difference in language level affect the level of difficulty of the item? This question cannot be answered without empirical evidence on how different item wordings affect performance.

This study examined the French version of the cognitive tests administered to students in Grade 8 as part of the Second International Mathematics Study by (1) having qualified people check the accuracy of the translation, and (2) performing statistical analyses of the achievement results.

Data Source

The data used in this study were drawn from the data pool of the Second International Mathematics Study (SIMS). The Population A tests (Grade 8) were administered to a random sample of 130 schools in Ontario, 115 English and 15 French. There were 180 items divided into a core form and four rotated forms. The core form (40 items) was administered to all the students. The rotated forms (35 items each) were ramdomly assigned to students within a classroom, with approximately equal number of students responding to each form. Thus, each student answered 75 items. For technical reasons 6 of the 180 were dropped from the analysis. The remaining 174 items covered five broad topics: (1) Arithmetic (58 items), (2) Algebra (31 items), (3) Geometry (42 items), (4) Probability and Statistics (17 items), and Measurement (26 items). All the items were five-alternative multiple choice (one correct response and four distractors). Every response to each item was coded into one of three categories: correct, wrong, or item omitted.

Method and Results

1. Accuracy of the Translation

The 174 items were examined by six bilingual mathematics educators, each working independently. A translated item was considered biased if at least two judges thought that it differed in some way from the corresponding item in English. The differences were classified in one or more of the following six categories:

a. differences in level of language difficulty,

b. presence of mathematical terms that are correct translations, but nevertheless different from the terms used in the students' textbooks,

c. differences in level of abstraction due to phrasing,

d. differences in the intended meaning,

e. differences in emphasis and/or in clarity of notation, and

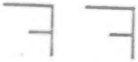
f. minors errors in translation (e.g., typographical errors not affecting meaning)

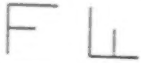
1.1. Examples

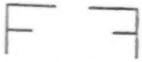
The item reproduced below, for example, was answered correctly by 79% of the students responding to the English version of the test, but only by 25% of those responding to the French one. A look at the item reveals many differences between the two versions: in the French version there is no mention of *pairs* of figures; the English text makes it clear that there is *one* correct answer, the French does not, the English text uses both *reflection* and *(flip)* whereas the French text refers to symmetry only.

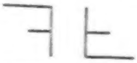
In which diagram below is the second figure the image of the first figure under a reflection (flip) in a line?

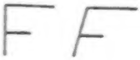
Parmi les figures suivantes, quelles sont celles qui sont symétriques par rapport à une droite?

A


B


C


D


E


The following comments on items are additional examples of the kind of disparities in the translation:

- **Item 29 in Form 4. (error category: a, c; percent correct: English 32, French 23)**

  The English item includes the familiar words "turn", "flip", and "slide", in addition to the technical terms "rotation", "reflection", and "translation". The French item uses technical terms only.

- **Item 7 in Form 0. (error category: c; percent correct: English 70, French 42)**

  The English item contains the explicit and detailed sentence "shows a cardboard cube which has been cut along some edges and folded out flat" versus the very formal "développement d'un cube". Also the second sentence "which two corners will touch corner P" is much simpler than the French "quels points seront confondus avec le point P".

- **Item 11 in Form 3. (error category: d; percent correct: English 31, French 16)**

  In this case the meaning was changed completely in the translation. "Turn left and move one unit" is not equivalent to "On fait un pas à gauche" (which is more likely to mean simply "step to the left"). The correct answer to the English version is A (1,-2) whereas the correct answer to the French one is C (0,-1).

- **Item 23 in Form 3. (error category: d; percent correct: English 43, French 23)**

  Response B in French is entirely different from the English one. B is an incorrect response in the English version of the item, but a correct one in the French version. Thus the item has two correct responses in French, instead of one.

- **Item 27 in Form 4. (error category: e; percent correct: English 26, French 14)**

  The variables *a*, *b*, *x*, and *y* are in italics in English but not in French. For this reason it is difficult to understand the French item. Also in the English version the clause "including the deposit" is underlined, whereas it is completely missing from the French version.

The above are a few of the striking examples of disparities. In addition, many items used relatively unfamiliar words in their French version: e.g., "consommé" vs "eaten" in item 4 Form 0, "juxtaposés" vs "put together" in item 18 Form 2, "se transforme par rotation" vs "can be rotated (turned)" in item 32 Form 3.

In sum, the examination by the six bilingual teachers revealed that in the French translation, a total of 70 out of 174 items (about 40% of the test) differed in some way from the corresponding items in English.

2. Statistical Analyses

The achievement data of the two groups were examined in two different ways. The first was an investigation of the rate of omitted responses while the second was the use of the transformed item difficulty approach to detect biased items.

2.1. Differences in Rates of Omission

The mean rate of omission over the 174 items was greater for the French students than for the English students, (7.1% vs. 2.8%). On the average, the omission ratio of French to English was 2.5:1. The rate of omission in itself is not proof of item bias, since students omit items simply because they do not know the answer, but it is consistent with the lack of clarity observed in many of the item translations.

Table 1 shows the mean omission rates of the French and the English students by test form. The t-tests paired comparisons indicate that with the exception of Form 2 all the differences between the two groups are statistically significant at the .01 level.

Table 1

Mean Percent of Omitted Responses and t-values of Differences between English and French Groups

Form	English	French	t-value
0	2.37	7.80	-7.19*
1	3.06	8.94	-5.89*
2	2.79	3.32	-1.13
3	2.61	6.66	-7.20*
4	3.27	8.48	-6.53*

*p< 01

2.2. Transformed Item Difficulties

Identification of biased items is problematic, since the indices available for this purpose are notoriously unreliable (Linn, Levine, Hastings and Wardrop, 1981). It has also been shown that these statistical indices of bias are essentially uncorrelated with judgment of item bias (Hoover and Kolen, 1984). An attempt was nevertheless made to explore item bias in the data through statistical means, using the technique of Transformed Item Difficulty (TID). An item is considered biased when it is comparatively more difficult to answer correctly for one language group than for the other. The two sets of correct p-values, one for each group, were transformed to normal deviates (z) by reference to a table of the normal curve as suggested by Angoff and Ford (1973), and then to Delta values ($13 + 4z$). The bivariate graph of the sets of Delta values then shows the degree of dispersion of the items. Clearly, the more the correlation between the two language groups on their responses to the items deviates from a perfect correlation ($r_{FE} = 1$), the greater the dissimilarity of their response patterns in each group. The measure of group-by-item interaction, that is, the magnitude of item bias, is represented by the perpendicular distance of any particular item from the major axis line (the line that minimizes perpendicular distances).

Table 2 shows the means and standard deviations of item deltas for the two groups on each test form. The means for the French students are higher than those for the English students, indicating that the tests were more difficult for the French students. The evidence that item-by-group interactions are present in the data comes from an examination of the correlations between the item deltas. The correlations are well below 1, ranging from .93 for the Core test (Form 0) to .81 for Form 1. The interaction of item-by-group stems from at least two factors: (a) the rank order of the item difficulty for English students is not the same as that for French students, and (b) there are relative differences in item difficulty,independent of rank ordering. Either factor would indicate item bias.

Table 2

Mean Delta Values, Correlations, Slopes, and Intercepts
by Test Form

Form	MeanF	StdF	MeanE	StdE	Correl	Slope	Inter
0	14.34	2.27	12.32	2.07	.93	1.11	.71
1	14.86	2.25	12.94	2.33	.81	.96	2.47
2	14.71	1.89	13.48	1.91	.83	.99	1.41
3	14.63	1.65	13.08	1.92	.84	.84	3.67
4	14.56	1.98	12.70	2.09	.89	.94	2.61

2.3. Identifying Biased Items

Figure 1 shows the plot of delta pairs (for English and French students) for test form 1.

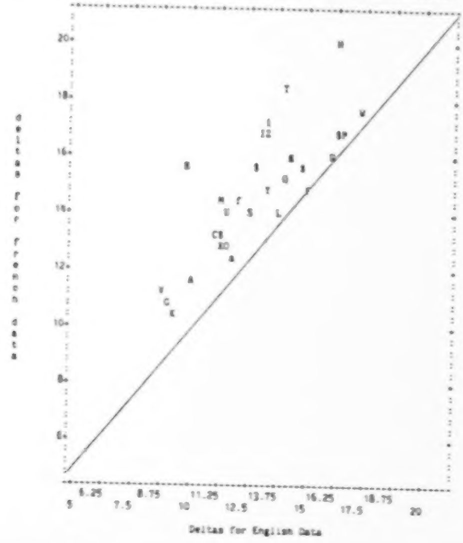


Figure 1. Delta Plot French vs. English - Form 1.

An examination of the plot reveals that the items are more difficult for the French students. (This in itself is not an indication of item bias; it could be due to a difference in ability or curriculum between the two groups.) The plot also reveals that some items deviate considerably from the line of best fit, and thus contribute a great deal to the lack of internal consistency of the test. These deviant items may possibly be sensitive to external factors that do not influence the other items in the test. The statistical analysis cannot identify these external factors, of course; they could be differences in curricular emphasis, curricular content, or translation, among others.

Far fewer items appear to deviate from the line of best fit than were identified as biased items by the judges. On the other hand, some of these deviant items are clearly not mistranslated, but rather appear not to have been taught to the French group (e.g., item T). Others could in fact have been deviant because of difficulties due to poor translation (e.g., item B shown on page 2).

In sum, there was a poor correspondence between the biased items detected using the TID method and those identified as biased by the judges. This may not be surprising, in view of the known limitations of the statistical method. Though limited in its capacity to identify specific items, the TID analysis did give an overall indication of item heterogeneity.

Summary

A statistical analysis of the achievement data of English and French Ontario students showed that disparity of translation might have introduced a systematic error in the measurement process, making a valid cross-language comparison of achievement very difficult if not altogether impossible.

This study has shown differences in level of difficulty between the two versions of the SIMS Grade 8 achievement tests. These differences may have been caused in part by the discrepancies between the English and French versions of the tests; any comparison of achievement between these two Ontario language groups must take this into account.

It is reasonable to expect that similar differences among other SIMS countries in the wording of their test would also lead to spurious differences in achievement, and thus any international comparisons of achievement should be concerned with the possibility that such disparities might be found in the other nine translations of this test.

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PEER INTERACTION IN A PROGRAMMING ENVIRONMENT

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INTRODUCTION

Mathematics educators have recently turned their attention to the role of discussion and peer collaboration as aids to pupil learning. Although it is reasonable to conjecture that 'talking' (in both its cognitive and communicative functions) and listening (in an active way) generate increased understanding and facilitate integration of previously fragmented context specific knowledge, actual research on peer collaboration effects has been sparse. Such supporting evidence that is available tends to be within a Piagetian framework and concerned with the notion of cognitive conflict. Another way to achieve a shared task perspective is however to assume complementary rather than conflicting roles. It has been suggested that the work of Vygotsky offers insights into the intellectual value of inter-peer support particularly with regard to 'scaffolding' the learning task (Wood, Bruner and Ross 1976) in order that a partner might achieve a 'level of potential development' rather than a level of 'actual development'. In any exploration of these ideas it is however important not to ignore the possibility that collaborative work (especially over an extended period of time) might impede individual acquisition of particular domains of knowledge and skills as pupils come to rely on their peers to achieve particular goals.

While most educators agree that the microcomputer has the potential to promote interaction among pupils, there has again been little systematic investigation of the dynamics of the learning groups; that is the individual responsibilities assumed, the kinds of interaction occurring and the effects of these interactions. Two of the aims of the Logo Maths Project (Hoyles, Sutherland & Evans 1985,see footnote to Sutherland & Hoyles in this proceedings) were to investigate:

1 the nature and extent of collaboration between pupil pairs learning Logo and differences in the collaborative patterns between pupil pairs.
2 the influence of discussion between pupil pairs of on the "efficiency" of their problem solving strategies and their understanding of programming and/or mathematical ideas.

OVERVIEW OF RESULTS IN RELATION TO PEER INTERACTION

1) There is no doubt that the computer provided an engaging problem solving context. It was evident that not only were pupils provoked to talk but also that a large proportion of the talk was task related in contrast to other research).

2) Our observations lead us to believe that pupils tend to have a 'natural' style in their computer work (which seems to be gender related). This varies along the following dimensions:

Careful planning ----------------Open ended investigation
Focus on global characteristics -----Focus on local detail and on immediate graphics or text output or on overall mental plan
Systematic --------------------Not systematic

3) Despite marked variation between the patterns of interaction between pupil pairs, instances for each pair have been recorded when collaborative exchanges have:

- provided challenging ideas for projects
- kept the project going in the face of "obstacles"
- changed the level of representation of the work (conceptual to concrete or vice versa)
- provoked reflection on the process within a procedure to predict its outcome.

4) We have found that pupil pairs tend to have implicitly negotiated individual dominance for particular aspects of the activity. This negotiation of dominance has impeded individual acquisition of particular understandings in some cases.

5) For all the case study pairs the amount of pupil talk has increased markedly over the three years of the research; there are more exchanges before decisions as to the action are taken and the exchanges are longer. Gender related differences in the nature of the pupil utterances are observable; in particular girls tend to more consistently refer to their partner within the utterance rather than refer to the task.

CODING OF PUPIL UTTERANCES

A classification system for the pupil discourse has been developed in order to obtain an overall picture of the qualitative nature of the peer interactions, to facilitate comparisons between the pupil pairs and to monitor changes in interaction patterns over time. All the verbal "on task" utterances of each pupil during their Logo activity have been coded using the categories given in Table 1. A pupil utterance was delimited by either an utterance of a partner or a specific action on the computer. Random extracts of transcripts were coded independently by two researchers and a coefficient of reliability of between 80 and 85% obtained.

For the purposes of this paper reference is made to the codings of one pupil pair (Sally and Janet). The data will be presented at the conference. It shows that most of the pupil talk was specifically action oriented; that is focussed on 'getting the task accomplished'. Little attempt particularly at the initial stages of the Logo work, was made by the pupils to explain or convince one another of what was meant or why a proposed course of action should be taken (a feature observed for all the case study pairs and particularly marked for boys). A qualitative development over time in the language of the pair was observable with a general move towards more elaborated argument and explanation. Despite this overall trend it is apparent that other factors crucially influence the nature of interaction; in particular:

- the nature of the task (real world representation or abstract)
- the extent to which the task is locally or globally planned
- the extent of assymetric negotiation and dominance of one partner
- the extent of explicit (or even implicit) agreement as to overall strategy or plan
- the extent to which pupils are 'getting on' socially

A COMPARISON BETWEEN INDIVIDUAL AND COLLABORATIVE WORK

In addition to the overview analysis as described above, description and analysis has been undertaken of collaborative interchanges which, together with computer feedback, have played an important role in the gradual modification or reorientation of a pupil's conception of a mathematical or programming idea (as identified in an individual setting). This analysis is ongoing. One example is given below:

Individual work

Each pupil was given individually the "Lollipop" task as shown in Fig. 1. This task was designed to investigate:

a) whether the pupils individually were able to build a general procedure with variable input
b) how the pupil coped with the internal relationships assigned within the task
c) whether the pupil perceived modularity within the task (i.e. whether a square procedure was used).

For the purposes of this paper we will focus on aspect b). We were interested in whether the pupil's would ignore the internal relationship given, employ an "additive" strategy for inputs (that is introduce a new input for each different part of the structure that 'varied' (see Hoyles 1986)) or make the relationship explicit by the use of a scalar operation on one input. Sally and Janet, though working individually, exhibited very similar programming styles; they both wrote separate start up procedures, worked initially in direct mode recording their commands and introduced inputs at the point of building a procedures using a 'substitution' strategy (that is replacing specific inputs to commands by named variables). Both girls using an 'additive' strategy for their variable inputs as can be seen from their final programs (Fig. 1). In addition both girls experimented with their procedures choosing inputs which did not represent the ratios in the figure (is this yet another example of how children seem to circumvent the situations we present to them?!)

Collaborative Responses

One week later Sally and Janet were given the "Arrow" task (Fig. 2.) to work on together. Again we shall focus here on how the pupils working collaboratively coped with the internal ratios given. The discussion was recorded, coded and transcribed. The coding of the pupil utterances indicates a high overall level of elaborated argument and discussion. (24% of utterances at level 2 or 3). A more detailed analysis showing three distinct phases in the pair's work is given in Table 2. These phases exemplify the pupils preferred working style. Firstly there is a quick 'try out' in direct mode (Phase 1). Here utterances tend to be procedural and there is a large proportion of hands on activity. The figure produced in direct mode by the pair did NOT reflect the ratios given, AB was equal to BE. (It is interesting to speculate as to whether this is related to the pairs action orientation and lack of reflective speech in this phase.). Phase 2 follows in which there is detailed discussion of how a procedure should be built. Utterances are more elaborated especially when there are disagreements over the plan. We give some details below of the role of this discussion in relation to the way the pair decide to deal with the internal ratios in their figure.

The girls spontaneously see that they can use the same input for AB and BE but want at the outset to use another input for DE.

J *Alright --- we work it out cos that will have to be something called JACK that JILL and that JILL if you get what I mean ---- all the 50's then the 25's*

They start to build a procedure:

```
  HILL 'JACK 'JILL
  RT 90
  BD :JACK
```

At this point Sally intervenes:

S *Wait a minute you have to do --- no BD MUL ---*

She wants to operate on an input and tries to elaborate why.

S *--- Em you say for this one you say BD :JACK and for this one you multiply it by two cos thats hal.*

Joanne does not understand and disagrees

J *But you have to put in another number---*

Sally seems to become more confident in her idea in the face of Janet's conflicting perspective. She tries to justify her proposition and in so doing provides Janet with some 'scaffolding'

S *But we're not going to put any old number in cos it won't be the same pattern--------that's what I'm saying.*

Janet then begins to see the point

J *I don't know how to do it---we could get rid of JACK?*

In Phase 3 the girls enter their procedure definition. Their utterances became task focussed. Janet made the final decision to eliminate one input.

This episode shows how the pair achieved collaboratively a sophistication in their use of variable that they had not thought to use individually.

A Further Phase

The screen output in this case did not provoke Sally and Janet to see that their figure did not 'match' the ARROW given. A specific intervention was point to ask them to check the lengths. The way they coped collaboratively with this new more complex situation (to make the lengths in the ratio 6:5:3) will be discussed at the presentation.

FINAL DISCUSSION

It is evident from this paper that the role of peer interaction in a computer environment involves issues which are extremely complex. The particular episode described here does show a 'move' by both the pupils concerned in the area predicted as a result of the peer and computer interaction and illustrates how the computer environment can build 'scaffolding' for the learning task as well as provoke conflict. It seems that Sally was helped forward by articulating her thoughts, Janet by arguing with her and listening to her explanations. Observation of screen output were also important.

It is by no means certain however that pupils will make learning gains in tightly specified circumstances after collaborative work or discussion. It is difficult therefore for a teacher (or researcher) to predict with any precision what a pair jointly or individually will gain in any collaborative setting. We have examples of positive and negative collaborative effects which were completely unpredicted (and unpredictable!). Despite this we are certain that all the pupils have gained something from their partnerships --- at the very least independence of the teacher!

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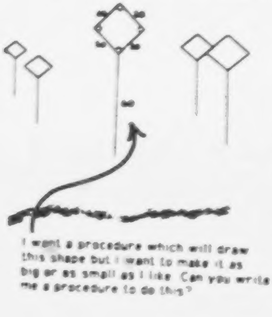


I want a procedure which will draw this shape but I want to make it as big or as small as I like. Can you write me a procedure to do this?

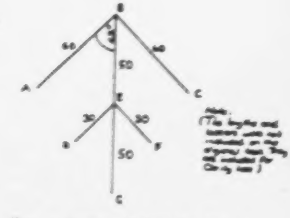
```
Sally s Program.

STICK "SIDE1 "SIDE2
FD :SIDE1
RT 45
REPEAT 4 FD :SIDE2 AND LT 90

Janet s Program.

KITE "VT "HT
RT 45
FD :VT
RT 90
FD :VT
RT 90
FD :VT
RT 90
FD :VT
RT 90
FD :VT
BD :VT
RT 90
FD :VT
RT 45
FD :HT
```

Figure 1
The Lollipop Task



Make a Procedure To Draw This Shape As Big or As Small As You Wish

Figure 2
The Arrow Task

Table 1 CATEGORIES OF DISCOURSE ANALYSIS

CATEGORY 0	CATEGORY 1	CATEGORY 2	CATEGORY 3	CATEGORY Q
On task discourse which has nothing to do with the heuristic.	a) Directions, Prompts suggestions, disagreements propositions with: (i) No Support (ii) Authoritarian Support (iii) Support with reference to the teacher	a) Directions, Prompts suggestions, propositions with simple support that is: (i) Support at local level or levels of action with focus on graphics or text output (ii) Support with reference to a specific example	a) Directions, propositions or explanations; (i) With reference to or indication of a global plan (ii) Including a second and a third proposition (iii) Which refer to a general rule, theory or pattern	Questions: a) Specifically requesting information
	b) Reaction to the screen with no explicit interpretation	b) Directions prompts suggestions, which form a second proposition	b) Disagreement with reference to or indication of (i) An alternative global plan (ii) An alternative general rule theory or pattern	b) Asking advice about a future plan or strategy
	c) Simple answer to direct closed question	c) Disagreement with reference to (i) Support at a local level (ii) A specific example (iii) An alternative proposition	c) Extension of the task which takes into account the global overall structure of the whole project	c) Requesting an explanation
		d) Directions for a simple extension to a task at local level only		
		e) Reaction to screen explicitly using screen information for support		

TABLE 2. TASK ANALYSIS FOR ARROW (SALLY + JANET).

		NUMBER OF UTTERANCES			NUMBER OF COMPUTER COMMANDS
		CATEGORY 1	CATEGORY 2	CATEGORY 3	
Phase 1	Drawing Shape in Direct Mode	18	3	2	21
Phase 2	Planning of Procedure Definition	28	6	7	0
Phase 3	Entering Variable Procedure	5	2	2	24

FACTORS AFFECTING SMALL GROUP PERFORMANCE IN PROBLEM SOLVING

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There has been considerable interest recently in students learning in groups rather than individually. Some of this interest is due to increasing emphasis on co-operative rather than competitive learning, but there is also a view that group activities have cognitive as well as social benefits. There are, however, issues about the efficacy and workability of such groups and how their outcomes can be assessed that have not been fully investigated.

In examining these issues, the authors have attempted to bring together aspects of educational-, social-, developmental- and cognitive psychology. Hopefully, what will emerge will be a framework which will allow pupils freedom to develop their ideas through discussion, negotiation and experimentation, whilst providing teachers with an adequate structure within which to evaluate pupils' development and progress.

This study is being undertaken as a new initiative in an established programme - the Assessment of Performance Unit (APU) Mathematics Monitoring Project - which monitors the mathematical performance of 11- and 15 year olds respectively in England, Wales and Northern Ireland. From national surveys, we have fairly comprehensive data collected using various types of written tests and from interviews, during which specially trained teachers asked individual pupils to perform tasks and give reasons for their methods. Data from the assessment of small groups of pupils are being sought to complement and enrich these findings by, for example, providing information on pupils' performance in groups as compared with performance when working alone.

In the above mentioned surveys, we have assessed affective and attitudinal components of both written and practical work on an individual basis. Now what we are seeking to assess now is the broader interface between mathematical problem solving (with any preconceptions that that engenders) and social interaction (with all that that implies).

Specific questions centre round the role of language in facilitating (or otherwise) mathematical problem solving, the optimum size and composition of groups which most encourage and maximise mathematical performance, the nature of mathematical situations that engage pupils' interest, how preconceptions affect approaches to the problem and how style of questioning and teacher interaction modify behaviour.

An assessment framework is being developed which attempts to describe group behaviour in a useful way. Broadly we are looking at 4 areas - social interaction, working on the task, mathematics used and communication of outcomes. More specifically, these areas are broken down as follows:

1. Social interaction - general features of group interaction,
   - are members of the group competitive or co-operative;
   - is there a dominant individual or do all members play an equal part?

2. Working on the task
   - how is the problem formulated, negotiated, resolved, extended?
   - are pupils involved, enthusiastic, persistent, etc?

3. Mathematics used
   - processes used - conjecturing, generalizing, systematic working, etc.
   - content areas - number, measures, etc.

4. Communication
   - explaining and justifying choices to an assessor or other children
   - recording
   - constructing

At present groups of 2-4 pupils are being asked to work on a variety of mathematical problems - some embedded in familiar contexts, some more evidently mathematical. Different group compositions are being tried - some based on friendship, others on teachers' assessments of pupils' ability and others on gender.

The study is still being developed, so no conclusive evidence can be offered; in fact, at this stage, we are raising more questions than we are answering. What is clear though is that, for most pupils (and many teachers), this type of approach to mathematics is unfamiliar and the conventions need to be negotiated and learnt. Once this has been done, initial results suggest that the outcomes can be positive and exciting for both parties.

More specific data on the issues mentioned in this synopsis will be discussed during the presentation of this research report.

NVCDs ARE STRUCTURING ELEMENTS

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Cognitive psychologists in general, and psychologists interested in mathematical education in particular, study the way children solve problems as well as the influence of the way these problems are presented to the child. In order to do so, they try to communicate with the child. They generally use the classical verbal language and ask questions such as : "Why did you do this ?" or "What do you think about this situation ?". Unluckily, this verbal language is not appropriate for such researches : it is too ambiguous and it is based on logical structures which are not obvious. Some researchers use a better technique to present a problem to a child and to **get a representation of what is happening in the child's mind** : they restrict themselves to logico-mathematical formal systems which have a rigid logical structure. By doing so, they lose some of the saddlety contained in the verbal language, but they gain in clarity and their results can thus more easily be interpreted. Unluckily again, the mathematical and abstract representations appear to be too cumbersome. We thus choose to use sets of tools which can be used as "a concrete logico-mathematical representation of (and isomorphic to) a formal system which is sufficient to perform reasonings". Such devices are called "**NVCDs**". Their properties have been described in previous papers. **COHORS-FRESENBORG**'s "Dynamical Mazes" (1978) can be used as an NVCD (LOWENTHAL, 1980, 1982). These mazes represent the mechanical equivalent of the hardware of a computer. They are presented to the children as railway-networks. Generally speaking, and NVCD is any set of tools provided with technical constraints : these constraints make certain actions possible and others impossible, and in turn suggest a logical structure.

Clinical observations were made while using NVCDs in classroom settings. These observations showed that NVCDs can be used by teachers and researchers to present a logico-mathematical problem to a child in such a way that the child understands it, while this would not necessarily be the case if the problem had been presented otherwise. NVCDs also enable children to build a solution which the child is often unable to produce otherwise. Teachers can use NVCDs to introduce new mathematical concepts (i.e. translations, symmetry, recursion, ...) and psychologists can use NVCDs to observe in an unambiguous way the behaviour of children confronted to problem solving activities.

These clinical observations enabled us to formulate the following hypothesis : "Providing children with an NVCD introduces a structuring element in the perception of data. One might assume that the introduction of such a device into a child's universe serves as a starter for a complex cognitive process". We assume that this complex process consists of 6 steps : 1) structuration of the perceptive field (in function of existing pre-concepts) ; 2) discovery of the relevant elements ; 3) building of local relations between some of these elements, implicit formulation of the relevance of such relations and experimental verification ; 4) shift in the level of relevance and construction of global relations between all the relevant elements, implicit formulation of the relevance of such relations and experimental verification; 5) verbal formulation and proof of a law concerning the theoretical functioning of the elements previously considered ; 6) transfer of the knowledge and of the structures acquired with NVCDs in a mathematical setting to other domains where NVCDs are not used.

We tried to prove this hypothesis in an experimental setting. As NVCDs are mathematically oriented tools, we could not use the progresses made in mathematics as a measure of the usefulness of our devices : this would introduce a systematic bias and we would be bound to have interferences between the technique used (NVCDs) and the topic which was taught (mathematics). Moreover, as far as our sample of 1st graders was concerned, they had been more exposed to some mathematical instruction than to any instruction concerning reading before they started to go to school. The use of scores concerning reading skills and their acquisition was thus bound to introduce less bias as far as prior knowledge of the children was concerned. We thus choose to use the scores the children obtained for a test concerning reading skills as a measurement of a possible transfer of knowledge and structures acquired during mathematical activities to other domains.

EXPERIMENTAL SETTING

The Dynamical Mazes were used as NVCD with 6-year olds in a laboratory setting. The children's progress was compared by measuring their progress in reading. These children belonged to two different classes (class A and class B, both 1st grades) and they had two different teachers (teacher A and teacher B). The children were tested before treatment using INIZAN's predictive test for reading (BP) : this enabled us to divide both groups into an experimental group and a control group in such a way that inside a given class, the two subgroups were equivalent as far as INIZAN's predictions concerning reading were concerned ; the two classes were not equivalent : the average score of class B was significantly better than the average score of class A (STUDENT t test, p = 0.016). The two teachers did not use the same teaching method as far as reading was concerned : teacher B asked the pupils to formulate hypotheses, to test them and to adapt them while teacher A used a more systematical and more conventional method. INIZAN's evaluation test of reading performances (BL) was used as post-test.

TABLE 1 : PRETEST (BP) AND POST-TEST (BL) SCORES

	CLASS A		CLASS B	
	BP	BL	BP	BL
CONTROL	47	24	54	23
	40	28	60	32
	50	26	50	24
	38	25	51	40
	59	34	59	40
	45	27	54	38
	53	37	55	33
	55	46	52	46
	46	46	58	26
	58	54	50	44
	50	52	56	43
	57	53	48	46
			57	43

	CLASS A		CLASS B	
	BP	BL	BP	BL
EXPERIMENTAL	50	28	55	55
	60	32	59	57
	39	34	51	55
	42	30	46	49
	47	39	50	52
	49	36	60	45
	51	42	54	48
	54	44	60	42
	49	48	55	42
	58	49	58	41
	57	44	53	34
	57	58	57	32

TREATMENT

The children of the experimental groups worked by groups of 2 with the Dynamical Mazes. During a first session, they freely manipulated the material ; during the remaining 6 sessions they were asked first to build a maze with the given material corresponding to a small sketch and then to discover its use : "the first train will leave the maze through gate A, the next one through gate B, and the 101st train through gate ...". It was thus possible to establish whether these children learned to make short, medium or long term prediction. This fact was important since the reading method used by the teacher of class B was based on such predictions. A child did not work constantly with the same child, but the children of the experimental groups A and B never worked together.

TABLE 2 : MEANS AND STANDARD DEVIATIONS

CLASS A	Predictive Battery (pretest)		
	Control	Group : M = 49.83	SD = 6.59
	Experimental	Group : M = 51.08	SD = 6.20
	Reading Battery (post-test)		
	Control	Group : M = 37.66	SD = 11.37
	Experimental	Group : M = 40.33	SD = 8.49
CLASS B	Predictive Battery (pretest)		
	Control	Group : M = 54.15	SD = 3.63
	Experimental	Group : M = 54.83	SD = 4.14
	Reading Battery (post-test)		
	Control	Group : M = 36.77	SD = 7.98
	Experimental	Group : M = 46	SD = 7.80

TABLE 3 : CORRELATION COEFFICIENTS (BRAVAIS-PEARSON)

Class A	Control group	.616 (Significant)
Class A	Experimental group	.430 (Not significant)
Class B	Control group	.237 (Not significant)
Class B	Experimental group	- 0.245 (Not significant)

RESULTS

A between-means STUDENT t test was used to perform inter-classes, inter-groups and inter-groups intra-classes comparisons.

1. Inter-classes.

   Class B is significantly better (p = .016) at the pretest. There is no difference at the post-test.

2. Inter-groups.

   There is **no** difference at the pretest, but the experimental group performed significantly better at the post-test (p = .015).

3. Inter-groups intra-classes.

   There is no difference between control and experimental group of a given class at the pretest. As far as the post-test is concerned, there is no difference inside class A but there is a very significant difference in favour of the experimental group inside class B (p = .005).

4. There is only one case where the correlation between predictive and reading battery is significant : the control group of class A.

DISCUSSION

1. The results concerning the correlation coefficients suggest that either the teacher of class B is unusually good or that the use of logically structured material with children aged 6 favours their acquisition of reading skills. The other results show that both factors are present in this study.
2. The data shows that it is important for the pupils to have a good teacher, **familiar with a method based on the production by children of hypothesis which must be tested and adapted.** This "mathematical" attitude is clearly useful in domains very different from mathematics, since pupils with lower prediction-scores, score as well as others for the final reading-test.
3. The better score of the global experimental group shows that the use of NVCDs with very young children seems to favour their cognitive development in general, and to structure their learning activities.

CONCLUSION

The finer analysis "inter-groups intra-classes" shows that NVCDs are not very important when a logically structured teaching method is used, whatever is the subject of the lesson. Conversely, when such a logically structured method is not used or is not available, or when the children are rather weak, NVCDs constitute logically built structuring elements which can serve as excellent complement, even for typically non mathematical topics.

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Examining the Heuristic Processes of Nine to Twelve-Year Old Children in Small Group Problem-Solving Sessions

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Studies of children's mathematical problem-solving behavior have illustrated the effectiveness of student's working in small groups to solve problems (Alston & Maher, 1984; Maher & Alston, 1985; Noddings, 1985). Results suggest that the processes exhibited by children working in small groups are similar to those used by children working individually. The interplay among the individuals in small groups including the proposal of alternative ideas and their representation often enhances the effectiveness of the problem solving. In an earlier paper (Maher & Alston, 1985) both group and individual behavior were the unit of analysis. The study indicated that generally group behavior reflected the individual behavior comprising it although some individuals never quite became part of the group.

Goldin (1984, 1985a) has described cognitive representations as including the features of planning and executive control. For planning, the heuristic process is given as the unit of analysis. In another paper (1985b) he defined categories of the subprocesses and provided a detailed prototype script as a model for the study of the heuristic process, "think of a simpler problem" (TSP). Maher and Alston (1985) adapted the prototype designed for use with individual children for use by groups in a study of seventeen very able twelve-year old children. This report used the adaptation to include younger children and a broader range of mathematical ability.

PURPOSES

The study had several purposes: (1) to observe the consequence of group work; (2) to determine whether and how homogeneous grouping according to grade-level and ability (to the extent possible) contribute to successful problem solving; (3) to determine whether and how children working in groups could (a) apply the heuristic TSP, (b) recognize a pattern and (c) construct the solution; and (4) to determine whether children who had successfully constructed a solution could generalize to problems of equivalent structure.

DESIGN

Subjects

The subjects were 61 children: 27 from grade 4; 21 from grade 5; and 13 from grade 6. All were members of one of four sections of a problem-solving analysis class taught by the same instructor. The mixed (4-6) class was part of an enrichment program for academically talented children. Criteria for admission were: (1) scores in the upper 5% of their class and testing in the 95th percentile in an area of a locally administered standardized test, or (2) participation in a school program for talented children; or (3) having a strong teacher recommendation for the potential to achieve (1) or (2). The children came from 52 communities in New Jersey and were divided approximately equally between boys and girls.

The Problem Task

The group problem-solving task used was the same as in the 1985 Maher/Alston study. However, information gained from the earlier study directed attention to the behaviors of children that fell into three of the original five parts. These include observing children's behavior as they (1) addressed the problem posed, (2) responded to the suggestion in the script that they "make a chart showing the remainders in order when 2 to some power is divided by 3", and (3) were asked to generalize the solution to $2^{44}/3$, $2^{75}/3$, and $3^{50}/4$.

PROCEDURES

Two consecutive 75 minute periods were provided for completion of the task. Five groups of two children and 17 groups of three were formed according to grade level and ability to the extent possible. For the seventeen three member groups, six were comprised of 4th graders; three of 4th/5th graders; three of 5th graders; four of 6th graders. For the five two member groups, one was made up of 4th graders, two of 5th graders, and one each a mixed 4/5 and 5/6 combination.

Ability was subjectively determined by the instructor's rating of children's performance according to five criteria: (1) participation in class, (2) performance of homework assignments, (3) flexibility in thinking, (4) verbal statement of ideas, and (5) persistence in seeking solutions. Children frequently meeting most of the criteria were classified as level A. Those sometimes meeting most were classified as level B, and those rarely meeting them were classified as level C.

Data came from audio and video tapes of the children's behaviors, observers notes, and children's written work.

RESULTS

A group was designated "successful" if it arrived at a correct strategy and reached a correct solution. None of the 22 groups were successful initially or spontaneously generated the pattern in the remainders. After responding to the suggestion that a chart be developed for successive powers of 2, nine of the groups successfully generated the pattern and used it to arrive at a correct solution. Of the 9 groups, 8 were three-member and one, two-member. Of five two-member groups, the successful two-member group had the oldest children and the highest teacher rating. Two groups had conflicting solutions to the problem in which one or more members recognized the pattern and generalized the solution but at least one member disagreed. Two groups were successful in constructing the solution by recognizing and using the pattern of remainders; however in the effort to generalize the mode of solution to $3^{50}/4$, the children reverted to earlier misconceptions about the structure of exponents. Nine groups were unsuccessful in constructing the solution.

Age and grade level seemed to be related to children's success. Of the six three member fourth grade groups, only one was successful in constructing and generalizing the solution. The two member fourth grade group was also unsuccessful.

The results were consistent with the instructor's rating of children's ability. None of the four groups comprised of children with C ratings was successful. The five successful three-member groups included at least one child rated A and no children rated C.

The nature of interaction among group members indicated a relationship to the success of the group effort. Observers noted that in four of the successful groups, interaction was critical to solving the problem. In one case, for example, two of the children assisted the third in arriving at a solution. The children themselves reported that working as a group was the most important element in solving the problem. Active cooperation was particularly important in the group analysis of their first solution of two as the remainder when $2^{50}/3$ because 2^{10} times 5 divided by 3 gave that remainder. Later, a second member reached the same solution of 2 because 2^{5} times 10 divided by 3 gave the same remainder. The three girls compared the two solutions and questioned the result since the value of 2^{50} was not the same in both cases. Such critical questioning prepared the group to analyze the information in the chart and conclude that the solution arrived at by this analysis was more reliable.

Observers also noted that a principal impediment to successful group action was the incapacity for accepting for consideration the proposals of others. In one group of three 4th graders there was a child intent upon his own method of direct computation of 2^{50}. He did not respond to the effort of one of the girls to explain the pattern and her conclusion from it that the remainder was one. He insisted on his method rejecting the logic of the girl's proposal. During the disagreement, the third member was silent and inactive.

Success appeared to depend on two factors: previous experience with exponents and the realization that direct computation of 2^{50} involved a long series of multiplications by two. Of the group comprised of younger children, all except one were unsuccessful. They had no previous experience with exponents. The one successful group had had this experience. Yet of the nine successful groups all but one began their solutions with attempts to compute 2^{50}, which they later abandoned.

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6. LOGIC AND PROOF

"Illegal Thinking" in Solving Geometric Proof Problems

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Many investigations, concerning human learning and thinking, especially problem solving, have been restricted to deliminated problem classes. Restrictions of this kind ensure a rather small range of suitable strategies, which can easily be overlooked, but do not exclude that modifications will be necessary in order to cover all strategies occurring in any other special problem field.

A universal approach to describe and analyze human problem solving within the framework of cognitive psychology is the idea to define a class of problems as a set of possible objects and a set of operators being used to transform these objects in successive order, starting from an initial object (the givens) and achieving the final object (the goal) by a sequence of intermediary objects (intermediary goals).

An operator consists of a premise or a list of premises, and a conclusion, thus, allowing to preceed from the further to the latter after having checked whether the givens or any already established object match the premise(s) of the operator.

Heuristic strategies select the operators, assess their effectiveness, compare the intermediate goals already achieved with the final goal. The model of information processing in human learning and thinking basically assumes a well-defined set of operators, but in so-called synthesis problems even the means, i.e. the operators, have to be invented, at least partially (Dörner, 1979, p. 14). Thus, it is not surprising that human subjects tend to invent new operators even in problem situations where the acquired operator repertoire would be sufficient.

Dörner (1973) has established a system of categories of "illegal thinking", based on proofs of theorems in propositional calculus. He describes the following types of "illegal thinking", illustrating them by erroneous problem solutions taken from the mentioned topic area:

(NCON) Non-consideration of the conditions for the application of an operator
(NEX) Non-consideration of the application instructions
(AN) Invention of new illegal operators by analogy transfer
(SEM) Invention of new illegal operators by "semantic" considerations
(PAR) Invention of new illegal operators par force
(EXT) Search for external causes for the "unsolvability" of a problem.

The types (PAR) and (EXT) also seem to occur in many other topic areas, and so does the random trial-and-error strategy, which is not enumerated in the quoted list.

The author has systematically analyzed seventh and eighth graders' errors in geometric proof problems since 1981, and could ascertain the use of the types (PAR) and (EXT), the latter being comparable to a haphazard trial-and-error strategy. The type (EXT) in the propositional calculus problems is based on algebraic analogies preferably, in other topic areas analogies of logical propositions could be observed.

Since in propositional calculus operators reveal a rather simple structure - in comparison f.i. with geometry -, the types (NCON) and (NEX) are extremely comprehensive and therefore have to be modified and specified when being applied to other topic areas, such as proofs in plane geometry, due to the far more complex tasks.

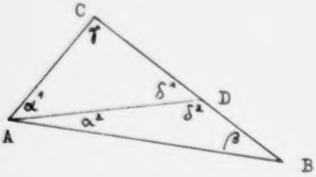
Solutions of proof problems performed by seventh and eighth graders show
(1) lack of identification or erroneous matching of variables in the premise of an operator
(2) erroneous application of an operator - which corresponds to (NCON), above -
(3) incorrect identification of variables in the conclusion
(4) erroneous execution of an operator - which corresponds to (NEX) in that application of (4) not only omits certain symbols which a correct application would produce (cf. the original, Dörner 1973), but generally has as outcome an alteration of the correct conclusion

(5) confusion of premise and conclusion of an operator.

In the following passage one typical example for each category is given. The examples are translated from German, without considering mistakes in the German text. The author's comment is marked by [ ].

(1) Lack of identification or erroneous matching of variables in the premise of an operator

Context: To prove: If in a triangle $L(\overline{AC}) < L(\overline{BC})$, then $W(\beta) < W(\alpha)$.



Solution:
Auxiliary line AD,
by which $L(\overline{AB}) = L(\overline{BD})$.
Isosceles triangle theorem:
$W(\alpha^2) = W(\delta^2)$.
...

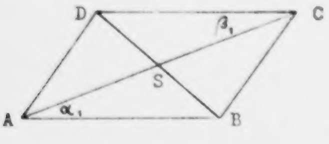
[Notice that in the quoted passage the proof is formally correct. But the premise about the equal length of the sides $\overline{AB}$ and $\overline{BD}$ is not kept to. Thus, the correspondance between the sides $\overline{AB}$ and $\overline{BD}$ in the figure and the denotation sides of equal length in an isosceles triangle is not correct.]

(2) Erroneous application of an operator

Example: The well-known application of a congruence theorem which actually does not exist, such as "aaa", or - before having dealt with it - a congruence theorem running "ssa", without checking, whether the mentioned angle is the one opposite to the longer of both sides.

(3) Incorrect identification of variables in the conclusion

Context: To prove: The diagonals of a parallelogram bisect each other.



Solution:
Since opposite angles are equal,
according to theorem 15,
the triangles ABS and DSC are congruent to each other.

For they have as corresponding parts

$\alpha_1 = \beta_1$

$\overline{AB} = \overline{DC}$

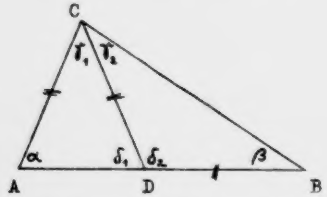
and angles S .

...

[The quoted theorem is the following: In a parallelogram opposite angles (!) have equal measures. Confer the notion "opposite angles in a parallelogram" with the meaning in the present context.]

(4) Erroneous execution of an operator

Context: To prove: If in a triangle $L(\overline{AC}) < L(\overline{BC})$, then $W(\beta) < W(\alpha)$.



Solution:

[The signs in the figure obviously indicate equal length of the marked sides.]

$W(\gamma_1) = W(\delta_1)$ [obviously application of the isosceles triangle th.; lack of direct quotation of applied theorems]

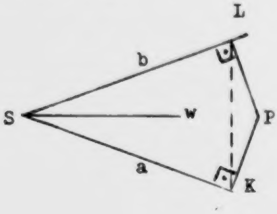
...

$W(\alpha) > W(\delta_1)$

...

(5) Confusion of premise and conclusion of an operator

Context: To prove: If a point has equal distances from the sides of an angle (of measure less than 18o°), then it lies on the bisector of the angle.



Solution:

After having drawn a line through K and L, we obtain two isosceles triangles SKL and PKL.

An isosceles triangle always has two equal angles and two equal sides.

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"IF..., THEN..." STATEMENTS REVISITTED

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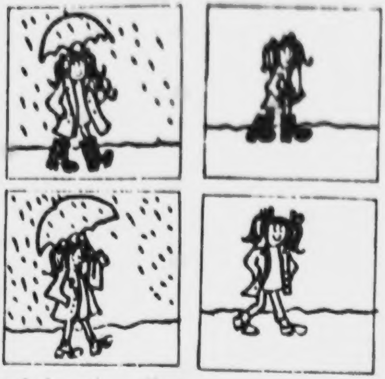
Research Goal

In the previous decade the study of children's logical thinking took a major place in educational research. In most cases syllogistic reasoning was at the focus of these studies, examining students' ability to judge validity of conclusions drawn from simple premises which include a conditional statement. This line of research was deserted in the eighties leaving many questions unanswered. In particular it was not clear whether non-valid conclusions were drawn due to disability to distinguish valid from non-valid inferences, or due to misinterpretation of the conditional premise. The main goal of this study was to uncover various interpretations, or rather misinterpretations students assign to conditional statements. 160 students at age 11, 14, and 17, participated in the study.

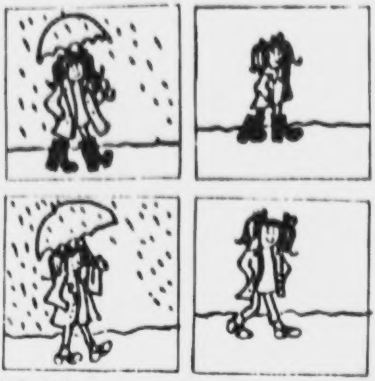
The Instrument

Each student was presented with two sets of 16 tasks in a written form. The second set was taken a fortnight following the first one. Each one of the 32 tasks was of the following format: Given 4 pictures related to one verbal statement of the form: "If p, then q", identify and cross off the picture or pictures which disagree with the statement. The given statement was typed verbally and communicated a reasonable content. The 4 pictures related to it represented "p and q"; "p and not-q"; "Not-p and q"; "Not-p and not-q". The two sets consisted of 16 corresponding tasks. In each pair of corresponding tasks the same four pictures appeared, but the statements were two converse conditional statements, i.e. "If p, then q" in one set was replaced by "If q, then p" in the corresponding task in the other set.

For example, here is a pair of two corresponding items (see correct solution at the end of this paper):



If it rains, then
Ruth has her boots on.



If Ruth has her boots on,
then it rains.

In the rest of this paper "^" designates "and"; "-" designates "not"; "→" designates "If ..., then ... ".

Main Research Questions

We analyzed the answers according to the following questions:

a. To what extent can students at age 11, 14, and 17 recognize the picture representing "p and not-q" as the only one which disagrees with: "If p, then q"? In other words, what is their ability to interpret "p→q" as logically equivalent to: "-(p^-q)" (namely, contrary to "p and not-q") or to "p^q or -p^q or -p^-q" (namely, ""p and q" or "not-p and q" or "not-p and not-q"")?

b. Is there a pattern to the wrong answers within age group? In particular, do wrong answers tend to fall into one specific category amongst Piaget's 16 binary operations? E.g. if students largely tend to cross off the two pictures representing "p and not-q" and "not-p and q" as disagreeing with "If p, then q" this would imply that they interpret "If p, then q" as meaning the same as "p^q or -p^-q", which would mean interpreting a conditional statement as a biconditional.

c. If there is a pattern of misinterpretations to conditional statements, does it change through age, and how?

d. Is sex difference evident in this process?

Main Results

As expected, analysis of the results revealed a significant increase with age of the ability to give the correct answer (mean age-group score 26.8%, 48.5% and 66.6% for age 11, 14, 17 respectively). It is, however, noteworthy that only the oldest group exhibitted a fair control of the meaning of conditional statements. No sex differences were found with this respect.

The analysis of wrong answers was more surprising. For each age group, the most frequent wrong answer was different, yielding a kind of a "developmental path" on the road to the right meaning of conditional statement without evidence to sex differences:

At age 11 the most frequent wrong answer was leaving "p and q" as the one and only option which does not disagree with "If p, then q", by crossing off as disagreeing with "If p, then q" three out of the four given pictures. In other words this age group largely considered a conditional statement as logically equivalent to a conjunctive one. (See an example at the end).

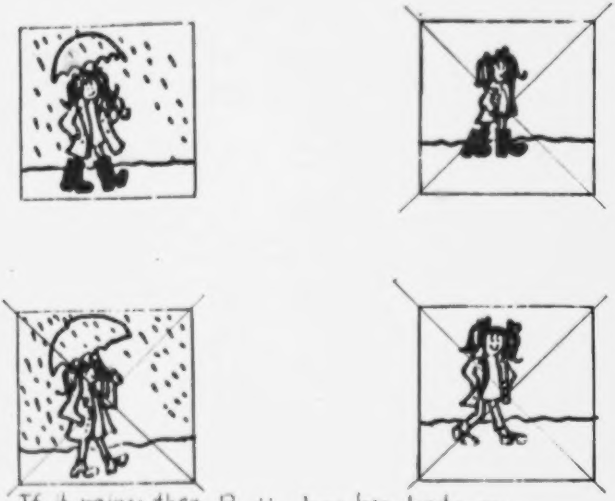
At age 14 the two pictures describing "p and not-q" and "not-p and q" were most frequently crossed off as disagreeing with the corresponding conditional statement. This means attributing "If p, then q" the same meaning as "p if and only if q", not distinguishing between a conditional statement and its inverse. (See an example at the end).

At age 17, "If p, then q " was interpreted wrongly most frequently as "p or q". This was exhibitted by crossing off just one option (unfortunately the wrong one): "-p and -q", leaving as equivalent to "If p, then q" the pictures: "p^q or -p^q or p^-q" which is logically equivalent to "p or q".(See an example at the end).

A Brief Discussion

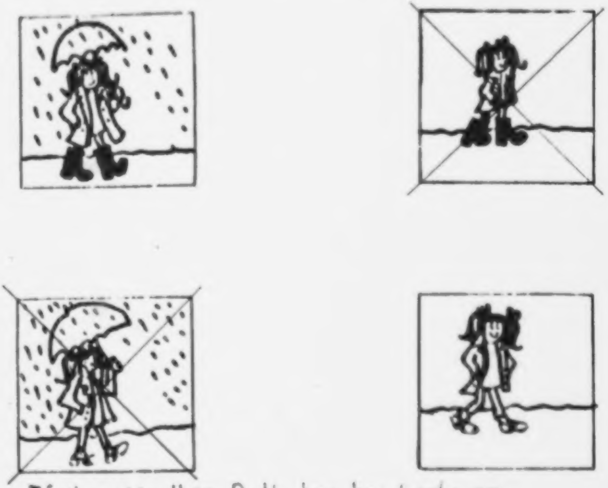
This study indicates that the understanding of conditional statement develops gradually through various levels getting "closer and closer" to the correct one, through sort of a descending series of misinterpretations. This may have an impact on students' performance in processing information of a hypothetical-deductive nature such as they are expected to carry out in school mathematics and science.

An example of a typical wrong answer at age 11



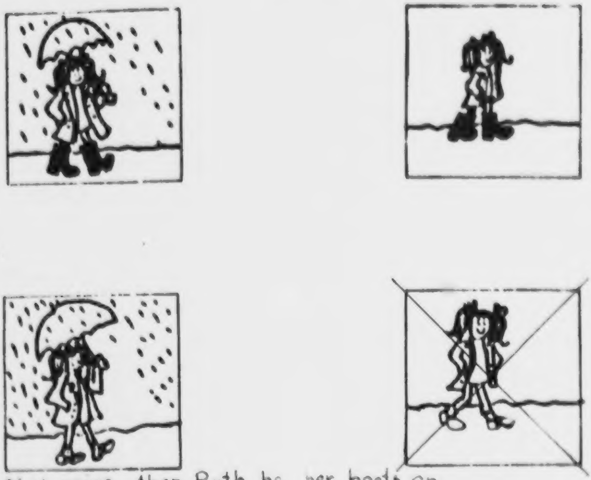
If it rains, then Ruth has her boots on

An example of a typical wrong answer at age 14



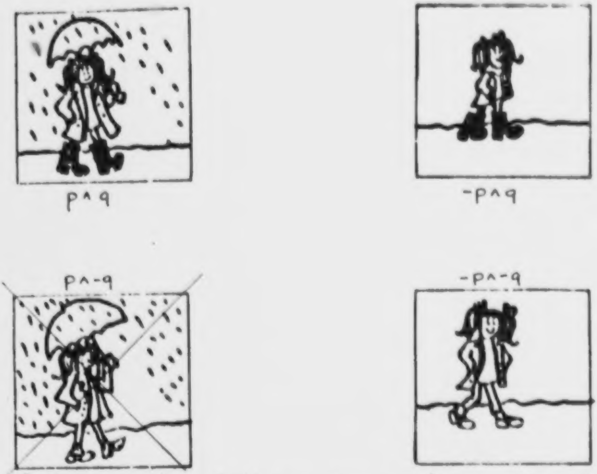
If it rains, then Ruth has her boots on

An example of wrong answer at age 17



If it rains, then Ruth ha her boots on

The correct solution



$p \wedge q$

$-p \wedge q$

$p \wedge -q$

$-p \wedge -q$

If it rains, then Ruth has her boots on ($p \rightarrow q$)

THE CONCEPT OF PROOF HELD BY PRESERVICE ELEMENTARY TEACHERS:
Aspects of Induction and Deduction

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Recent research has appeared exploring the important topic of students' concepts of proof. Fischbein and Kedem (1983) focused on the question of whether high school students understand that a mathematical proof requires no further empirical verification. Vinner (1982) focused on the question: What makes a sequence of correct mathematical arguments, a mathematical proof in the eyes of high school studnets? We focus on a different population, the prospective elementary school teacher. Since proof receives very limited attention in elementary school text books, the main source of experiences with verification and proof is the classroom teacher. It is therefore important to understand the conceptions of proof held by elementary school teachers. We further focus on a somewhat unique aspect of proof, related to inductive and deductive reasoning.

Theoretical Framework

Anderson (1980) distinguishes between an "inductively valid argument" (IVA), an argument whose conclusion is not necessarily true, but only probable, and a "deductively valid argument" (DVA), an argument whose conclusion must be certain (if the premises are true). One can then view the sequence of mathematical arguments used to prove a mathematical statement as a DVA, of which the mathematical statement is a certain conclusion.

The viewpoint that a mathematical proof must be a DVA is certainly held by mathematically sophisticated persons. On the other hand, our experience suggests that lower-level mathematics students often have a different point of view; namely that an IVA can be a proof. A psychological rationale supporting our impression follows. People in every-day life form or evaluate hypotheses by estimating the probability of the hypotheses with respect to their relevant individual experiences. (See hypothesis formation and hypothesis evaluation in Anderson, 1980.) Relevant individual experience includes evidence that supports or refutes the statement whose validity is in question. It seems likely that this behavior pattern is translated in the mathematics classroom by lower-level students to acceptance and production of IVAs as proofs for mathematical statements; the students accept and provide examples as a legitimate process of mathematical proof. Furthermore, this viewpoint may be reinforced by instruction at lower levels, which frequently uses examples to verify mathematical statements.

As these students encounter higher mathematics, at the high school and university level, instructors present DVAs as mathematical proofs and stress (at least implicitly) that IVAs do not constitute mathematical proof. Our question, then, is what conclusion the students draw about the role of IVAs and DVAs in mathematical proof. Do students accept the point of view that only DVAs constitute mathematical proof? Do they continue to accept IVAs as proof, even of mathematical statements?

Methods

The views of proof held by 101 students enrolled in a mathematics content course designed for preservice elementary school students were assessed; two sections of the course were offered, taught by two different instructors. Our assessment was based on responses to a test in which seven verification-types of mathematical statements were offered. Verifications of two mathematical statements were presented in separate sections; one was discussed within the mathematics course, while the other was unfamiliar. Here we discuss only the students' reactions to the familiar mathematical statement, with respect to inductive and deductive reasoning. The familiar statement follows:

> "If the sum of the digits of a whole number is divisible by 3, then the number is divisible by 3."

Students were asked to rate each verification on a scale of 1 to 4, where 4 indicated that a verification is considered a mathematical proof, and 1 indicated that a verification is not considered a mathematical proof at all. A summary of the presented verifications follows.

Example-type Verifications

Examples. Two particular situations, using small numbers, in which the statement was shown to be correct.

Pattern. A chart giving a sequence of numbers for which the statement is true, along with determination of the truth of the condition and conclusion of the statement.

Big Numbers. A big number for which the statement is shown to be correct.

Contrapositive. A specific example supporting the contrapositive of the statement, and a specific example supporting the original statement.

Deductive-type Verifications

General Proof. A correct and general proof of the proposition in the case of 3-digit numbers, including statements justifying each step.

False Proof. A fallacious general proof in which none of the steps in the inferential chain were correct.

Particular Proof. A correct proof of the proposition, including statements justifying each step. However, the proof is presented as the verification of a particular example, rather than as a general proof.

Results and Discussion

We now consider our original research questions.

IVA: We categorize students' views of IVA as follows. If a student rated any of the example verifications highly (i.e., as 3 or 4) then s/he considers some form of IVA to be valid verification of the mathematical statement, and we accordingly categorize such a student as **High IVA**. If a student rated any of these IVA-based verifications as 4, we categorized him/her as **Very High IVA**. On the other hand, we consider a student rating all of these verifications low (i.e., as 1 or 2) as being **Low IVA**; if a student rated all of these IVA-based verifications as 1, we categorized him/her as **Very Low IVA**. Note that Very Low IVA and Very High IVA are subclasses of Low IVA and High IVA respectively. Note also that the classes of Low IVA and High IVA are disjoint. The distribution of the categories is shown in Table 1. High IVA occurred much more often than Low IVA, by a margin of 86% to 15%; this difference is significant at the .001 level ($\chi^2 = 24.96$, df = 1). The difference is also striking when considering the extreme ratings of IVA; 64% of the students accepted some sort of IVA at a very high level, while only 4% rated all example-verifications very low.

Table 1. Frequencies of IVA levels

	High	(Very High)	Low	(Very Low)
Percent (N=101)	86	64	15	4

DVA: Students' views of DVA are categorized by considering their ratings of General Proof (GP). We categorize students who rated GP as 3 or 4 **High DVA**; those rating it as 4 are categorized as **Very High DVA**. On the other hand, those rating GP as 1 or 2 are categorized as **Low DVA**; those rating it as 1 are categorized as **Very Low DVA**. As with IVA, Very High and Very Low DVA are subclasses of High and Low DVA, respectively; High and Low DVA are disjoint. The distribution of categorizations of DVA is given in Table 2. Many more students are categorized as High DVA than as Low DVA, by a margin of 76% to 25%; this difference is significant at the .001 level ($\chi^2 = 12.88$, df = 1). 54% rated GP very high, as 4, while only 3% rated it low, as 1.

Table 2. Frequencies of DVA levels.

	High	(Very High)	Low	(Very Low)
Percent (N=101)	76	54	25	3

Relationship of IVA and DVA: In contrasting students' ratings of IVA and DVA, we make the following categorizations. Students who rated DVA high and IVA low, we call **Only DVA**; students who rated DVA very high and IVA very low, we label **Extreme Only DVA**. Students who rate IVA high and DVA low, we call **Only IVA**; students who rated IVA very high and DVA very low, we label **Extreme Only IVA**. Students who rated both IVA and DVA high, we label **IVA+DVA**; the case where both are rated very high, we label **Extreme IVA+DVA**. The remaining case, in which both IVA and DVA are rated low, is not of interest in the present study. The distribution of the categories is given in Table 3. Note that only 14% can be categorized as Only DVA, while 62% are considered IVA+DVA, with 24% Extreme IVA+DVA.

Table 3. Relationship of IVA and DVA levels.

	Only IVA	Extreme Only IVA	IVA+DVA	Extreme IVA+DVA	Only IVA	Extreme Only IVA
Percent (N=101)	14	3	62	28	24	2

We thus summarize our answers to the original research questions as follows:

1. Most students accept DVA as a means of verifying mathematical statements. This is not suprising since they were evaluating a verification which had been identified in class as being a proof. The following is more suprising.

2. Most students hold the IVA and DVA points of view simultaneously. Students do not feel accepting DVA excludes acceptance of IVA.

3. Most students accept IVA as a means of verifying mathematical statements.

4. Few students accept DVA as the only means of verifying mathematical state-

ments. This is consistent with the findings of Fischbein and Vinner that students need more than deductive proof to accept the truth of a mathematical statement; they need empirical verification.

Additional questions: These results evoke additional questions:

1. Does the pretense of proof play a role in students' judgment of a verification? We can address this issue by considering students' responses to the False Proof verification type. 2. How do students view proof presented in a particular case? Students responses to the Particular Proof verification type will allow us to explore this question. 3. Do differentiations take place among the example-verifications with students in the High IVA category? What differentiations occur in the "IVA + DVA" viewpoint?

While this paper deals with verifications of a statement with which the students were already acquainted, verifications of a statement with which the students were not familiar wer also presented. We thus pose: 4. How stable are these categorizations in an unfamiliar context?

Two different instructors taught this course. While the same syllabus was followed by each instructor and efforts were made to keep the instruction standard, some differences in emphasis with respect to verification and proof occurred. In particular, one instructor spent more time on proof within the context of geometry. Thus we ask, 5. Are the categorizations effected by amounts of attention to proof?

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CHILDREN'S APPRECIATION OF THE SIGNIFICANCE OF PROOF

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I. The problem

What is it that really convinces a pupil of the truth of something mathematical? Of course, in practice it is the teacher - if sir says so then that's that. But what if she has to decide for herself, without the guidance of an established authority? What sort of evidence is then the most convincing?

This question arises out of a wish to come to grips with what it is to understand mathematics. Much has been writted about understanding in our subject (see, for example, Skemp (1979)). For the purpose of this study understanding and knowledge will be taken to be essentially the same thing (though not literally so), and knowledge will be taken to involve three conditions (see, for example, Scheffler (1965)): for a person A to know X

(i) X must be true;

(ii) A must believe X to be true;

& (iii) A must have good reason to believe X to be true.

Teachers want pupils to come to understand many general statements, of the form "all P are Q". For a mathematician criterion (iii) involves proof, and so teachers, because of this, may wish to prove results. On the other hand criterion (ii) demands that children should genuinely believe what they are learning, and there is room for doubt as to how much they appreciate the significance of a proof and how much more convincing they find other types of evidence. Following Bell (1976), evidence will be taken to be of two kinds:

(i) One may check a variety of particular cases, and when sufficient examples have been checked become convinced that the statement is universally true. This is the empirical approach.

(ii) One may investigate the structure of the mathematics involved, analyse the relationships with other, already known, mathematics, and eventually prove the statement. This line, more mature mathematically, will be called the logical approach.

The purpose of the research reported below was to investigate children's autonomous uses of these types of evidence, and so to assess their grasp of the significance of proof. The research took the form of a narrative questionnaire and interviews. In the age-range 11 to 16, 390 boys and girls completed the questionnaire, and of these 50 participated in a series of three task-based interviews.

II. The Questionnaire

This took the form of a three part narrative with questions interposed at appropriate points. Three different mathematical topics were used. In each part Andrew and Peter were arguing about the truth of a statement, like "the sum of three consecutive whole numbers is always divisible by three". (Andrew was the one who was right in each case.) After empirical evidence was produced, the subject was asked whom he thought was right, and whether or not he was sure about this. The reasons for the judgement were also asked for. The statement was then proved by Andrew, Peter claiming that he still thought it might break down for some very big numbers, and the subject was again asked whom he thought was right. An analysis of the pre-proof and post-proof judgements gave the following results:-

(i) The pre-proof judgement. Nearly half of the responses given showed that the pupil was sure that Andrew was right. Of these, though, only 21% gave logical evidence for the truth of the statement. Thus, about 40% of the responses indicated a complete acceptance of a proposition with nothing more than empirical support, and in only one case in ten did a pupil produce a proof of her own. The proportion of judgements based on logical grounds was found to increase with age in a significant way ($P < 0.01$).

(ii) The impact of the proof. If a pupil does not succeed in providing logical support for a statement, then the next best thing, mathematically, is for him to acknowledge that he is unsure, and to be convinced only after reading the proof. Such a person appreciates the relative merits of empirical and logical support. Of those pupils who did not produce a proof of their own nearly one quarter fall into this category. Interestingly, while 26% of girls' judgements show this sort of appreciation of the significance of proof, the figure for boys is only 20%. This gender difference is statistically significant ($P < 0.05$).

III. The Interviews

It is widely recognised that as a tool for the investigation of the thinking patterns of children doing mathematics, the task-based interview can be extremely fruitful. Because it is interactive it can be flexible enough to allow the interviewer to develop ideas mentioned by the pupil and to probe particular points where this is helpful. For this study five boys and five girls in each of the five year groups were given three interviews, based on a series of related tasks in arithmetic, geometry, and "common-sense". An interview comprised from 8 to 10 items, each of which was a statement which the subject was to classify as true or false, giving reasons. The first three were 'dead' items (the subject did not know this) whose function was to familiarise the subject with the topic of the interview and help her develop confidence in her handling of the materials. There followed two true generalisations, one fairly easy and the other harder. Examples of the items used are:-

(i) Arithmetic. "Adding 3 to a number and doubling the result always gives the same answer as doubling the number and adding 6."

(ii) Geometry. A large square lattice board was used, the dead items ensuring that it was to be thought of as extending indefinitely. Difficulties which arose in the pilot study suggested that a games format would be most fruitful: for each item a few rounds of the relevant game were played before the item was presented. In "Hide and Seek" a single lattice point in the middle of the board was designated a tree. The interviewer chooses a lattice-point at which to stand, and the subject tries to find another lattice-point at which to stand where he will be hidden behind the tree. The statement to be judged is, "The second player will always be able to find a place to hide."

(iii) "Common-sense". A large square "checker-board" was used (again of indefinite extent) to represent the planning board of a landscape gardener. A supply of rectangular pieces of card, each able to cover two of the squares on the planning board", represented paving-stones. The subject was shown how the cards could be used to make paths, where the width was always one unit, and patios, where the width could be greater than one unit. It was explained that paving-stones could not overlap or be broken into two. The first, easier, statement to be judged is, "Any square path can be made."

Of crucial importance to the study was the use of "particular" items after each true generalisation. It was important to produce an objective test of whether or not the subject really did believe the true generalisation. So, the arithmetic item was followed by, "Adding 3 to 16 and doubling the result gives the same answer as doubling 16 and adding 6." Similar "particulars" followed all the true generalisations. In fact, as each "particular" was presented the interviewer pointed out that it was a particular case - the phrase "it is just the same as the last one" was used.

For each item the subject was asked to justify her decision to classify as true or false, and there was fairly full discussion designed to develop her ideas as fully as possible without actually giving any leads.

The following is an outline of the main results:-

(i) It was found that the justification given for the true generalisation was of a logical nature in about 15% of the cases. This is rather more than the 10% figure from the interview study, probably because of the supportive role adopted by the interviewer. There was again a significant ($P < 0.05$) upward trend of the proportion of logical responses with age.

(ii) Subjects were asked, after they had classified the true generalisation, whether they were sure, or not quite sure, that they were right. It was found that boys were significantly more likely to claim to be sure than girls ($P < 0.05$).

(iii) There were identified three responses to the "particular" items which followed their generalisations. They were deduced from the generalisation (classified immediately as true), they were checked as an individual case (the full working-out was done for the item, with no reference at all to the previous item), or they were checked as an embodiment of the proof. This last option was rare, since it was available only to those who had proved the generalisation. It consisted of the re-enactment, or rehearsal, of the proof around the data of the "particular" item. It represents a more sophisticated approach than simply checking as an individual case, but a less confident, more tentative approach than a straight deduction.

The important question is - under what conditions was the "particular" deduced? For only in this case can it be claimed that the subject really believes the generalisation. Does the deduction of the "particular" depend on the degree of sureness of the general, or on the type of justification for the general?

		Particular		
		Checked	Deduced	Proof re-enacted
General	Not quite sure	101	4	-
	Sure	139	39	7
General	Emp. support	228	19	-
	Log. support	12	24	7

It is clear that being sure of the generalisation is no guarantee of the deduction of the particular. We see, though, that 56% of logically supported generalisations are followed by deduced particulars, while a mere 8% of empirically supported generalisations lead to deductions. If we consider, further, that the seven "proof re-enacted" responses represent a sort of half-way stage between checking and deducing, reflecting a desire on the part of the pupil "just to make sure" that the proof is correct, then the evidence of the above table is overwhelming. It is the proof which counts. Pupils do not believe a general statement unless they have proved it.

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SOLVING MULTIPLICATION OPEN-SENTENCE PROBLEMS: THE INFORMAL DEVELOPMENT OF COGNITIVE STRATEGIES IN DIVISION SKILLS?

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Abstract

This research explored the mediating functions of solving open-sentence multiplication problems of the form $\underline{X \times U = Y}$ (Type 1) and $\underline{U \times X = Y}$ (Type 2) in learning to apply division algorithms. The processes that individuals may use in their solution are categorised into three models: Digital, Network, and Analog. Recent research suggests that the former two are more accurate representations of the likely processes employed in solving open-sentence multiplication problems. However, in terms of problem solutions by novice learners the first, the Digital model, was found to be the more appropriate explanatory model. Two experiments, one based on chronometric analysis the other on manipulation of concrete materials, designed to evaluate the use of three solution modes within the Digital model are reported. Educational implications which question the sequencing of concepts when teaching multiplication and division, multiplication generally being taught prior to division in contrast to simultaneous presentation, are discussed.

Background

Over the past decade several studies in Mathematics Education have generated a number of models which explain how children and adults solve mental addition, subtraction and multiplication problems. Little, however, is known about informal processes adopted by early primary children in solving simple mental division algorithms. Of three competing models, Digital (Parkman, 1971), Network (Ashcraft & Battaglia, 1978; Ashcraft & Stazyk, 1981), and analog (Moyer and Landauer, 1967), the former two have attracted the greater support as viable explanatory sequences for the processes which lead to solutions of mental arithmetic problems.

Early research studies (Parkman & Groen, 1971; Groen & Parkman, 1972; Groen & Poll, 1973) with samples of subtraction naive students suggested that for children, the digital model provided an adequate interpretation of reaction time (RT) data (chronometric analysis). The internal consistency of the Digital model, however, was questioned when explanations for multiplication problem solutions processes were being sought (Parkman, 1972; Stazyk, Ashcraft & Hamann, 1982) and recent research (Miller, Perlmutter & Keating, 1984) suggests that some form of retrieval process underlies performance of both tasks. Although this may be true for adults, it is more probable that such retrieval processes are not fully developed among novice learners. Ashcraft and Fierman (1982) demonstrated that the retrieval process is developmental and that in contrast to older children and adults, young children displayed significantly greater RTs in solving mental arithmetic problems.

In previous studies (e.g. Parkman & Groen, 1971) chronometric analysis was used to examine whether children solving open-sentence addition problems such as $X + U = Y$ (Type 1) or $U + X = Y$ (Type 2) where X and Y stand for given non-negative integers and U for the unknown, tend to do so by using an incrementing or decrementing process. Of three solution modes; $Z = Y - X$, $Z = X$, and $Z = \min(X, Y - X)$, in which Z represents the number of steps required for the solution of the unknown U, only the last, known as the MIN mode, has been found to produce a significant slope when observed RTs, T, are fitted to a linear regression function of the form $T = a + bZ$.

The object of the experiments reported in this paper is to demonstrate that in solving *multiplication* open-sentence problems, children who have not yet formally learned how to divide tend to employ the MIN mode as the operational process. Use of this mode of the digital model may indicate that young children have a more sophisticated awareness of division strategies than generally assumed. The three modes of the Digital model may be represented as: $Z = Y/X$ (incremental mode); $Z = X$ (decremental mode); and $Z = \min(X, Y/X)$ (MIN mode) where X and Y are the given quantities in multiplication open-sentence problems of the type $X \times U = Y$ (Type 1) and $U \times X = Y$ (Type 2) and U is the unknown.

All three modes were tested for both sentence types in two separate experiments. The first experiment was based on chronometric analysis and the second involved the use of manipulative materials. For each experiment, the sample consisted of fifty Grade 3 children, 26 boys and 24 girls, from two Catholic schools in Sydney, Australia. Each school had two parallel heterogeneous streams of students without special groupings to differentiate classes. The average age of the sample was 8 years 4 months (SD = 4.6 months). These children had formally been exposed to multiplication but no division.

Experiment 1

In order to present the problems to each child in a random fashion an Apple IIc microcomputer with appropriate software was used. The stimuli consisted of a total of 48 problems, 24 each of Type 1 and Type 2 which were presented in the form $X \times [\ ] = Y$ and $[\ ] \times X = Y$ respectively. Each subject was required to solve only the first 32 problems to appear on the screen so that each subject solved an equal number of Type 1 and Type 2 problems. Elapsed time was recorded.

Results

A series of regression analysis were performed on the mean reaction times, averaged over all subjects, to test the goodness of fit of each of the three proposed modes. Fit was determined by the standard F test for the significance of the slope parameter b. For Type 1 problems, the fitting of regression lines to the mean resulted in a significant slope $F(1,22) = 16.33$, $p < .001$ for the MIN mode. No significant slopes were found for the other two modes. For Type 2 problems, the fitting of regression lines to the mean resulted in two significant slopes. The MIN mode once again displayed a significant slope $F(1,22) = 13.78$, $p < .001$. Mode 2 ($Z = X$) also fit the data resulting in a significant slope $F(1,22) = 4.88$, $p < .05$.

Discussion

Of the three modes, the MIN mode best conforms to the hypothesised pattern of computational processes that might be undertaken by children who have not been formally taught to divide but who are attempting to solve multiplication algorithms in the form of open-sentence problems. Analysis of the data supports the face that the structural variable defined by the MIN mode provides the best account of the success latencies.

The familiarity of the subject with problem types may dictate whether this linearity manifests itself in the subsequent analysis. The fact that we found this to be true only for Type 2 problems was initially puzzling but further investigation suggested some viable explanations. Herscovics, Bergeron and Kieran (1983) indicated that most teachers tend to perceive a problem in the form of $M \times N = P$ as the number sentence "M sets of N" which is reinforced by a natural tendency to read from left to right (Kieran, 1977). Also, past research (Gunderson, 1953; Zweng, 1964, cited in Suydam & Weaver, 1970) has demonstrated the relative ease of Type 2 problems compared to Type 1 problems.

Experiment 2

A set of 24 3"x5" cards, each of which had one open-sentence clearly printed on it, were presented in two formats, 12 of each type; $X \times [\ ] = Y$ (Type 1); and $[\ ] \times X = Y$ (Type 2). Within each type one-half were identified as A-form ($X<Y/X$), the other half as B-form ($X>Y/X$). A set of "play-group" cards were available and used to illustrate problem solutions. Each card had 1, 2, 3, 4 or 5 "child-symbols" printed on it. The round smiling face symbol was used to depict a child. To illustrate their perceived solutions to a problem, the subjects were asked to place the relevant play-group cards in the appropriate rooms of a large flat cardboard "house" containing six rooms of rectangular shape and of the same size as the play-group cards. As the first card was presented, the researcher delivered a short story to place the problem in context and help the children in proposing solutions. For example, given the problem: $3 \times [\ ] = 12$, the solution would be the representation of the number sentence as four play-group cards, each with three smiling faces, placed into four rooms or three play-group cards, each with four smiling faces, placed into three rooms. For each open-sentence, the number of play-group cards selected was recorded.

Results

To test the assertion that Grade 3 children tend to be MIN oriented for multiplication-division open-sentence problems, an *a priori* test of the contrast of the average proportion of *X*-responses for *A*-form sentences and that for *B*-form sentences was performed for the Type 1 and Type 2 sentence formats separately using the Cochran's Q test (Cochran, 1950). Contrast estimates (w) and p-values were as follows: for Type 1, (w)=1.72 (p .01); for Type 2, (w)=2.18 ($p<.001$).

This analysis indicates that the proportions of *X*-results for *B*-form sentences tend to be much smaller than those for *A*-form sentences. The frequencies with which subjects selected *X* play-group cards for the Type 1 open-sentences also appear to be generally greater than for Type 2. The significance of this difference was tested by means of Wilcoxon's matched pairs signed-ranks test (Wilcoxon, 1945). The value of the test statistic was $T = 3$ ($p<.001$).

Discussion

The choice of number of cards was crucial because it was expected that number of cards chosen would reflect the known factor (*X*) for each open-sentence. The presence of MIN mode oriented subjects in the sample, however, would disturb this homogeneity because such individuals would tend to select *X/Y* play-group cards for *B*-form sentences. That is, they would choose the least number of cards. Given that the unknown was the MIN number exactly 1/2 of the time, then the expectation would be that through chance alone they would choose the MIN number for approximately 1/2 of the solutions. The results, however, indicated a significant and consistent use of the MIN mode.

Conclusion

Whether verified through chronometric analysis or illustrated through the use of manipulative materials, this consistent use of the MIN mode suggests some informal knowledge and use of division strategies by children untrained in solving multiplication open-sentence problems through the use of a division paradigm. It suggests that the subjects are well aware of and familiar with grouping and/or sharing concepts. On the occasions when they implement the reduction mode (X) they are in fact applying a division paradigm. Teachers should exploit this understanding by formally introducing the concept of division at the same time as they are introducing multiplication concepts. Intuitively it seems that this double edged approach may have benefits in attempting to help children understand the intricacies of division problems of a more complex nature.

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7. PROBLEM SOLVING STRATEGIES

Intercultural Studies between Indonesian and German Children on Algorithmic Thinking

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1. Introduction

Comparative intercultural studies in mathematics education are an interesting method to understand deeper the fundamentals of learning mathematics. We report on a pilot study as part of a research project in which the Centre for International Research in Teacher Education at the University of Osnabrück in W.Germany and the Department of Mathematics at the IKIP Sanata Dharma in Yogyakarta/Indonesia are involved. As part of a long term cooperation between our institutions the indonesian colleague had the opportunity for a PhD study (1982-86) at the university of Osnabrück included experimental studies in Yogyakarta (MARPAUNG 1986).

The field of our common research is the question, how children at early secondary level form algorithmic concepts, which are fundamental for the understanding of programing computers. Therefore the suitable methodology for our experimental studies is the use of clinical interviews with non-verbal methods in a standardized situation. As a consequence the size of the groups is rather small.

2. Remarks to the Indonesian Requests

Indonesia is a country which makes great efforts in economy to reach the technological standard which is necessary to give work for its increasing population. Therefore the educational system must grow, not only in size but in quality. Mathematics education including the understanding of computers plays an important role in this process.

Often it is said that using modern technology needs a specific kind of thinking which is more found in the western countries than in those of the Third World. We were interested to understand this assumption. A long term research project on the structure of algorithmic thinking at the university of Osnabrück was a chance for an intercultural comparative study.

Choice of subject

Important for the choice of our theme for the intercultural study was the following:

- the importance of teaching new technologies for our countries in the next decade
- the design of a nearly non-verbal approach to algorithms
- the happening of a similar study in Osnabrück.

Main Questions

The main questions of our research are the following:

1. How do indonesian pupils behave, compared with german ones when they are working on algorithmic tasks?
2. Which role play different forms of representation for the pupils while they are working on algorithms?
3. Do there exist differences in the thinking of the pupils while they are solving constructive or analytic tasks?
4. Which strategies and cognitive styles develope pupils in working with algorithms.
5. What is the role of strategies and cognitve styles concerning the performance of the pupils in solving algorithmic problems?

3. German Part

First pilotstudies on algorithmic thinking began at Osnabrück in 1981. An important aspect of the design (COHORS-FRESENBORG 1982,1983) was the possibility to represent an algorithm in 3 different forms. A detailed analysis of the relation between external representation of the algorithmic concepts and internal concept formation processes can be found in COHORS-FRESENBORG (1986).

Because of the fact, that language does not play an important role in this design, it was very useful for our comparative intercultural study. SCHWANK (1979) had first indicated, that using the networks of Dynamic Mazes enables the researcher to get in insight into the process of concept formation and problem solving of the pupil, because the sequential constructing of the network plays can be seen as a thinking-aloud protocoll.

The design of the intercultural study was developed by both authors in Osnabrück in close connection with the pilot studies, which were done by KAUNE to prepare her study. The indonesian colleague first paticipated in these pilotstudies with the german pupils, then he investigated the indonesian pupils. The german colleague vistited him in Indonesia during his main study and they both developed the final design there. Therefore the comparison between the german and indonesian pupils is not only based on similar tasks but also on personal experiences. The common design of both studies is reported in KAUNE (1985), a short survey see also SCHWANK (1986).

4. Experimental Studies in Indonesia

In a first pilot study in Indonesia 1983 we have choosen 12 pupils in the age of 13,5 - 15 of several Junior Secondary Schools in Yogyakarta. To compare them with the german pupils in one dimension we did the RAVEN-test.

The tasks of the german study had been translated to indonesian language, also the Registermachine-language. The problem solving sessions with the researcher were videotaped, so that the behaviour of the indonesian pupils could be analysed together with the colleagues in Osnabrück.

An important outcome of this pilotstudy was, that the indonesian pupils had great difficulties to play with the material Dynamic Mazes. They take this material in a very "stupid" way in their hands. Therefore we invented for the main study with the indonesian pupils one preparatory lesson in which they could become familiar with the material by constructing some automata-networks as they are described in COHORS-FRESENBORG (1978).

In the resulting main study in 1984 were investigated 14 pupils in the age of 13,5 - 14,7. The results in the RAVEN-test were between 6 and 39, mean 27,9, deviation 8,8. (The group of KAUNE had the results between 15 and 39, mean 28,4, deviation 7,1. The difference is caused by the one extrem result of 6 points.)

5. Results of the Indonesian Main Study

After only one hour of trainig with the material Dynamic Mazes the handicap, that indonesian children are not familiar to play with toys at home and with didactical material in mathematic lessons in school, even not at primary level, had no more great relevance. In flagrant oppositon to the pilotstudy the pupils could work with the Dynamic Mazes as it is known from the german ones and even some of them have choosen the Dynamic Mazes as the world in which they invented and analysed the algorithms. This shows, that the first observed cultural differences are important, but only on a very thin surface.

Contrary to the prejudice of the indonesian colleague before the experimental study has be done in Indonesia the indonesian pupils were as successful as the german ones in solving the complex algorithmic problems. There exist indonesian pupils who are as bright as german pupils. The kind of brightness during the problem solving process is very similar between the pupils of both culturs. The difficulties which had the indonesian pupils were already known from the german ones or have been found in further studies with german pupils. Only one essential difference could be seen in the beginning of the series of the 4 problem solving sessions: the indonesian pupils were obviously not as used as the german ones to solve problems by themselves and behave more in a way to wait for the comments of the researcher who must be very careful to let the pupil feel its own liberty for looking for ideas and in its decisions along the problem solving way. In our interpretation after long discussions this is the result of the strong authority of the adults and the appropriate teaching style in Indonesia.

Contrary to the results of KAUNE (1985) it makes no sense to distinguish the indonesian pupils by their success in constructive or analytic tasks (MARPAUNG 1986, p.67). But up to now the investigated group of children is to small for such a determination.

The cognitive styles of the indonesian pupils are as different as those of the german ones. For example the first 3 indonesian pupils in the rang scale by the RAVEN-test choose 3 different strategies of representing their algorithms (MARPAUNG 1986, p.55). The form of representation describes the world in which they form their concepts. This preferences for the use of represantation forms is stable: If a problem is given in a not prefered world the pupils' first action is to translate it into a form which is convenient for himself. Under the aspects of different cognitive strategies (COHORS-FRESENBORG/KAUNE 1984) the differences inside the group of german and indonesian pupils are more important than an intercultural comparison.

In the group of the indonesian pupils we could for the first time proof the hypothesis of SCHWANK (1985), that there exist 2 different cognitive structures in which algorithmic concepts are constructed: one is built by predicates (relations) the other by functions (operations). It could also be shown that the concepts "predicative/functional" in the sense of SCHWANK and "conceptual/sequential" in the sense of COHORS-FRESENBORG/KAUNE (1984) describe different phenomena. We have found one pupil who is working in a conceptual way with functional concepts, one in a sequential way using predicative concepts and of course the two other possible combinations (MARPAUNG 1986, p.98)

As a consequence it should be distinguished between the cognitive structure predicative/functional and the cognitive strategies conceptual/sequential (SCHWANK 1986).

6.Summary

In the beginning of our comparative study we believed, that there must exist cultural differences in algorithmic thinking. But the main result of our intercultural study is that the individual differences in the observed categories concerning the preferences for the form of external representation, the used internal cognitve structures and the followed cognitive strategies are of more importance insight the groups than between the both cultures.

We have recognized only one difference in the cultur of teaching, concerning the amount of educating pupils to creative problem solving behaviour.

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COMPARATIVE EXPERIMENTAL STUDY OF CHILDREN'S STRATEGIES WHEN DERIVING A MATHEMATICAL "LAW"

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THE PROBLEM

Eighteen pairs of grade 8 students in Australia and Japan were observed while co-uperatively searching for a mathematical "rule" in a geometrical context. The research was designed to provide answers to the following questions:

1. What strategies do students adopt when searching for a particular mathematical rule?
2. How do students react when they generate an example which contradicts a conjecture which they have advanced while searching for the rule?
3. How do Japanese and Australian grade 8 students compare on rule-search strategies and response to counter-examples?
4. To what extent do grade 8 students indulge in global planning when engaged in such activities?

All subjects were given the following problem:

> *A number of points are equally spaced around a circle. These points may be joined, in pairs, by line segments. The number of line segments depends on how many points there are. Find a method of determining the number of line segments needed to join all points. The method must work for any number of points.*

BACKGROUND

In Science, the justification for a proposition is generally achieved "through the accumulation of many supporting instances (or 'failures to disconform')" (Kemp, 1985). In Mathematics, however, the accumulation of such instances is not sufficient to allow the establishment of a rule. Instead, all cases must be considered if an empirical justification for a proposition is sought; survival of an empirical test of validity is not sufficient for the proposition to be considered 'true'.

Bell (1975) suggests that generalizing and proving activity among secondary-school children develops in four main stages. The first stage involves the recognition, extension and description of patterns or relationships. This is followed by empirical checking or attempts at deduction, as a second stage.

Awareness of the need to consider all possible cases develops during the third stage, while recognition of the need for explicit statements of the start of arguments and of definitions comes during the final stage. It seems that for children to engage in the final of Bell's stages, they need to be able to operate at the formal level. Collis (1975) suggests that only a small proportion of Australian children in the 13 to 15 age range display consistent formal reasoning abilities. It follows that most such children would not be expected to operate at Bell's final stage. Kemp (1985) believes that it is not until children have reached Bell's third stage that they begin to understand what constitutes a mathematical proof. According to Kemp, "[. .] it is clear that there is widespread misunderstanding about proof among adolescents and that [. . .] only a minority are able to produce mathematical proofs during their school years".

Balacheff (1985) identifies two phases which are involved when children engage in a law-deriving procedure: "one [that is] quasi-independent of the observer and one with strong observer-pupil interaction", the latter being a "rounding-off" phase directed by the observer. This process is similar to the classroom "elicitation pattern" described by Voigt (1985), which has three phases: a tentative phase, a phase in which the teacher focusses the students' discourse on the "official" solution of the problem, and a teacher-directed "rounding-off", phase. The approach of Balcheff's pupils may be likened to Voigt's second phase, refutations being produced by counter-examples generated by the students themselves rather than by the teacher. The third, "rounding-off" phase occurs in both schemes.

An advantage of studying pairs of children is pointed out by Schoenfeld (1983), who states that dialogue between subjects tends to encourage the articulation of managerial decisions, whereas in single-subject protocols such decisions are rarely overt. Balacheff and Laborde (1978) describe such subjects as being in a situation of 'social interaction'.

Schoenfeld (1983) points out that the metacognitive, managerial skills that experts bring to a problem-solving situation allow the entire solution process to be "watched and controlled, both at the local and global levels". Larkin, Dermott, Simon and Simon (1980) speak of pattern recognitions that "guide the expert in a fraction of a second to relevant parts of the knowledge store . . . [and which] guide a problem's interpretation and solution". It therefore seems unlikely that novices would exercise any substantial amount of global control of the problem-solving activity and would therefore tend to make few strategic, managerial decisions.

PROCEDURE

The sample consisted of 36 grade-eight children, working in pairs. Six pairs were from different classes in the same Australian boys' school and six were girls in the same Australian girls' school. In these cases, the observer was unknown to the subjects. The six remaining pairs were from the same Japanese class, the observer being their regular mathematics teacher.

Each pair of subjects was examined independently of the other pairs. In each case, one pencil and many sheets of paper were provided. Every word uttered and every action taken from the time the problem was given until the students stated that they had arrived at a final solution was recorded. The ensuing, "rounding-off" stage was not recorded.

TREATMENT OF THE DATA

The observations for each pair of subjects were initially transcribed into a protocol which was then parsed along the lines suggested by Schoenfeld (1983). The incidence of tactical and managerial decisions was determined from the parsed protocols. The data were then re-organized, flowcharts being constructed indicating the bases on which conjectures and predictions were founded and the strategies followed when a conjecture was supported or refuted as a result of being tested with data.

DISCUSSION AND FINDINGS

The protocols considered in this study correspond to Balacheff's (1985) first phase: that which is "quasi-independent" of the observer, which we earlier likened to Voigt's second phase. Examination of the data indicated that the protocols generated may generally be divided into two steps corresponding to Voigt's first two phases. Nearly every pair of subjects started with a lengthy phase in which they searched for a reliable method of constructing all the chords joining any given set of concyclic points. This was followed by a phase in which hypotheses were stated and (usually) tested as in Balacheff's (1985) study. The ensuing "rounding-off" phase, which invariably took place, was not recorded. In the present study, therefore, there were generally three phases which are similar to those in Voigt's classroom elicitation pattern.

Nine of the 16 pairs of subjects never encountered an example which refuted a conjecture which they had advanced. Five of the six Japanese pairs belong in this class. Four pairs of Australian subjects initially engaged in activities which eventually had to abandoned in favour of fresh approaches. During this intial phase, two of the above pairs either modified or abandoned the relevant conjecture, a third pair ignoring two refutations and abandoning the conjecture on the third occasion. During the successful phase which followed the false start, however, the above pairs responded to all refutations by modifying the data which produced them. In contrast, the fourth pair which produced a false start responded to refutations encountered in this initial phase by modifying the data which lead to the refutation.

The remaining pairs, one Japanese and three Australian, all encountered refutations during their rule-search procedures. The general tendency of each of these pairs was to modify the data leading to a refutation, although one Australian pair ignored one of its refutations and another modified the relevant conjecture in response to one of its refutations.

Many of the decisions made by the subjects were of a local, tactical nature. It seems, however, that the executive, managerial decisions made by children engaged in rule-search prcedures of the sort examined in this study can be divided into two categories which we label "projective" and "local". It is probable that only the former would be categorized as true executive decisions in Schoenfeld's scheme. Most pairs followed the initial reading stage with a projective managerial decision to examine a number of circles. This decision was then followed by one or more local, strategic decisions about which particular circle to examine. Two of the Japanese pairs made no decision that could be classified as managerial and one Japanese pair took the projective managerial decision to examine a number of circles only after a wild-goose chase and a second reading and assimilation of the problem. Apart from decisions to organize or re-organize the data, all other managerial decisions were of the local kind. Some were decisions to test a conjecture or a related prediction, followed by a tactical decision about how to do it. Others followed the refutation of a conjecture by a counter-example, and were presumably intended to advance the investigation in some way, sometimes by putting it back on the right track. There was a general absence of metacognitive, global management skills which experts use to control problem-solving procedures.

It is clear that none of the pairs of subjects was operating at a formal cognitive level. Many engaged in "naïve empiricism" (Balacheff, 1985, p 18). None engaged in a formal proof, nor did any explicitly recognize that all cases must be examined if an empirical test of the validity of a proposed law is to be established. Some pairs were content to claim that a pattern, or a sequence of numbers, from which the number of chords for any particular number of points could be determine, was a sufficient rule. Others were able to arrive at a more generalizable rule. The rules offered by subjects as solutions to the problem provided no evidence that grade-8 children operate at a formal cognitive level. Generally, subjects reached Bell's second stage of development for generalizing and proving activity, thereby supporting assertions made by Collis and by Kemp.

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COGNITIVE AND SOCIAL FACTORS IN PROBLEM SOLVING BEHAVIOUR.

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A number of researchers (Das et al.(1975); Hunt (1980); Ransley (1981)) have used Luria's (1973) model of brain function as the theoretical basis of factor analytic studies of cognitive abilities and their relationships to educational achievement. The studies have investigated individual differences in cognitive abilities as defined by three factors. The first two factors have been reported as defining two qualitatively and functionally distinct information processing abilities (Successive and Simultaneous Processing). The third factor has been reported as defining abilities to regulate cognitive processes (Executive abilities).

Crawford (1984) reported that initial results of a study of fifth grade students indicated significant socio-economic (SES) and gender differences in simultaneous processing abilities. There were no significant gender or SES effects associated with either successive processing or executive abilities. The results suggest that prolonged differences in experience (socialization) are associated with differences in performance in the visual/spatial tasks that are used to define the simultaneous factor.

According to Luria (1973), the two information processing modes are qualitatively distinct and fulfil different intellectual functions. A consideration of the functions of the two modes of information processing suggests some explanations for the apparent relationship between social experience and simultaneous processing abilities.

Successive processing is used to code information when the operational order is important as in imitative learning or following verbal instructions. It is not usually available for conscious introspection. Luria asserts that successive processing is important for operational aspects of cognition and the development of well established operational skills. Simultaneous processing is used, when immediate responses are inhibited (by means of executive abilities), to allow autonomous conscious reflection about the relationships between culturally salient attributes of the objective and social worlds. Luria suggests that simultaneous processing is used for higher order intellectual activity and is available for conscious introspection. Further, past experience in simultaneous processing results in concept development and an internalized frame of reference that guides future processing.

Research describing social interaction in the classroom (Fennema (1981); Spender (1982); Connell et al. (1982)) indicates gender and SES differences in experience. It seems possible that such differences may, over a period of time, affect the relative strengths of the two information processing abilities. Assertive male students from socio-economically advantaged backgrounds, with maximum similarity between the cultures of home and school, may be expected to have high abilities in simultaneous processing and to process mathematical information in an autonomous fashion with reference to internalized frames of reference. Low SES, female students with least similarity between the cultures of home and school and social expectations for less autonomous behaviour, would be expected to have low abilities in simultaneous processing, relative to successive processing. The choice of simultaneous processing at an effective level may not be available for the latter group. They may be expected to resort to help seeking behaviour and exclusively operational interpretations of mathematical tasks.

A Wilcoxin test was used to investigate gender and SES differences in the relative strengths of simultaneous and successive processing abilities, as defined by factor scores. The results are shown in Table 1 below. In each case a negative rank indicates that the factor score for simultaneous abilities was ranked lower than the factor score for successive abilities. The reverse situation is indicated by positive ranks.

The results support the notion that the qualitative and quantitative differences in experience described by researchers investigating classroom interaction are reflected in significant trends towards particular cognitive profiles. That is, for high SES males there is a significant tendency for higher abilities for simultaneous processing and lower abilities for successive processing. For females from the low SES group, there is a significant tendency for relatively high abilities in successive processing and low abilities for simultaneous processing.

GENDER	SES	-RANKS	+RANKS	Z	2 tailed prob.
GIRLS	HIGH N=23	N=13 MEAN 13.2	N=10 MEAN 10.4	-1.034	.301
N=52	LOW N=29	N=19 MEAN 17.6	N=10 MEAN=10.0	-2.541	.011
BOYS	HIGH N=34	N=9 MEAN 17.1	N=25 MEAN 17.6	-2.453	.014
N=70	LOW N=36	N=17 MEAN 17.2	N=19 MEAN 19.6	-.644	.519

Table 1. Wilcoxon tests of the relative ranks of factor scores for the simultaneous and successive variables.

For high SES females and low SES males there were no significant trends. One might speculate that the high SES female subjects experienced greater similarity between the cultures of home and school but social expectations for compliant behaviour. It is likely that low SES male subjects experienced greater social expectations for autonomous behaviour but less similarity between the cultures of home and school.

Observations of classroom interaction indicated that subjects with differing cognitive profiles interacted with classroom tasks in different ways. A more detailed analysis of problem solving behaviour was carried out using a subsample of 64 subjects. Multivariate analysis revealed a significant relationship between abilities for simultaneous processing and subjects' initial responses during the problem solving interview (see Table 2 below). Subjects were assessed according to their ability to understand the problem (1), express the problem in their own words (2), estimate a solution (3), select appropriate tactics to achieve a solution (4), and present the correct answer (5).

The univariate F values (not shown) for the relationships between the first four dependent variables and levels of ability for simultaneous processing were each significant. The variables were entered into the analysis in the order in which responses were required of subjects. The non-significant Stepdown F values for variables 2, 3, and 4 indicate that once demands on simultaneous processing abilities for initial interpretation of the problem were taken into account, demands for simultaneous processing resulting from responses required later during the interview procedure were not significantly greater.

N=64

SOURCE OF VARIATION	WILKS	APPROX MULTI F	STEPDOWN F VALUES USTAND (1)	EXPRESS (2)	EST. (3)	TACTICS (4)	CORR. (5)
SUCC.	.882	1.39	1.60	3.10	.38	1.17	.72
SIM.	.797	2.64*	10.26**	1.14	1.22	.72	.08
EXEC.	.860	1.69	2.45	1.12	3.25	1.53	.03
SUC X SIM	.960	0.43	.05	.26	1.01	.22	.65
SUC X EXEC	.975	0.27	.05	.13	.46	.73	.00
SIM X EXEC	.955	0.49	.67	1.41	.07	.28	.10
SIM X SUCC X EXEC	.870	1.56	2.93	1.21	3.38	.30	.02

Table 2. Summary Table of Multivariate Analysis of the Relationships Between Initial Responses and Cognitive Abilities. (*P<.05 **P<.01).

Halliday (1973) suggests that social experience and social contexts influence the role taken by subjects and the form and function of the language used. Subjects with different cognitive profiles discussed verbally formulated mathematical problems in different ways. The two cases described below are typical examples.

Vicki had high ability for successive processing and low ability for simultaneous processing and executive control. She read each problem accurately and without inflection. However she was unable to accurately express mathematical problems in her own words since the relationships between pieces of data were usually ignored. Simpler absolute statements were often substituted comparitive expressions. For example, Vicki was asked to describe the problem:

> Debbie has 26 pencils. David has 7 more pencils than Debbie. How many pencils do they have altogether?

She said ".....26 pencils, ...7 pencils .....you need to add them all up."

Nesher et al. (1982) would describe the above problem structure as involving a static comparison and a simple combine operation. Subjects with low abilities for simultaneous processing typically ignored the static comparison and interpreted the problem in operational terms. Vicki then discussed the meaning of the problem further with the experimenter. However, when asked to work out an answer Vicki ignored earlier reasoning and wrote: 26 + 7 = 33.

John had high ability for simultaneous processing and relatively low ability for successive processing. When asked to describe a problem in his own words he, and other subjects with similar cognitive profiles, described the problem context but not an operational strategy. When asked to describe the above problem he said:

> "Debbie has 26 pencils.....David has more....7 more...How many altogether?"

When asked to estimate a solution he said promptly:

> "It'll be two lots of 27 (gestures to indicate imaginary sets in different locations) and 7 more."

In the interview situation, subjects with high abilities for simultaneous processing usually responded with reference to the data presented in the problem. In contrast, subjects with low abilities in simultaneous processing, and relatively high abilities in successive processing tended to interpret the problem information in terms of the responses required by the experimenter. In Halliday's (1973) terms, the responses of the former group were typically "object-oriented" whereas those of the latter group were typically "person-oriented".

Conclusion.

The research reported above suggests that social factors influence the development and use of simultaneous processing abilities. Different cognitive profiles are reflected in the different ways in which subjects interact with problem solving tasks. Regardless of gender, subjects with differing cognitive profiles appear to have differing perceptions of the intellectual goal of tasks. These differences are evident in the form and function of the language used as well as in problem solving performance.

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EYE-MOVEMENTS OF FIRST GRADERS DURING WORD PROBLEM SOLVING

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1. Introduction

In our past investigations of young children's problem-solving processes with respect to elementary arithmetic word problems we collected empirical data mainly using individual interviews and the analysis of error patterns on paper-and-pencil tests (for an overview see De Corte & Verschaffel, in press). These studies have provided a rich set of specific findings concerning the appropriate as well as inappropriate information structures formed in representing word problems, and the variety and the development in the solution strategies used to solve them. However, until now, several other aspects of the problem-solving process, such as the text-comprehension variables and processes contributing to the construction of these information structures and the subject's decision-making processes while choosing a specific solution strategy, have received almost no attention.

Recently we have started to apply eye-movement registration as a new and complementary technique for generating and/or testing specific hypotheses concerning these latter aspects of children's solution processes on elementary arithmetic problems. In this contribution we present the design and some results of an exploratory eye-movement investigation undertaken during the last year.

2. Method

A series of eleven verbal problems were administered individually to a group of twenty high-ability (H) and low-ability (L) first graders near the end of the school year. The word problems represented eight different types from the classification schema of Riley, Greeno &

Heller (1983). With respect to each problem we collected eye-movement data while the child read and solved the problem, together with retrospection data in response to the question how (s)he arrived at the answer.

The eye-movement data were collected using the Debic 80 equipment, a German system based on the corneal reflection-pupil center principle (De Graef, Van Rensbergen & d'Ydewalle, 1985). The child was seating in a chair and the problems were presented on slides projected on a screen. While the child read and solved a problem, his eye-movements were registered every 20 milliseconds and represented in two different ways. First, the visual stimulus, together with the point the subject is looking at, were recorded on video; on the monitor the subject's point of regard was represented as the intersection of a vertical and a horizontal axes superimposed on the slide. Second, the coordinates of these subsequent intersections were stored on computer tape.

In view of analyzing the data, we had to define the elements or the areas of the perceptual field we were interested in. Therefore, we made a grid for each problem consisting of five horizontal and eight vertical zones.

The analysis was then done in two different ways. First, for each solution process we computed the total number of measurements and the percentage of the total solution time during which the subject was looking at each particular area; these gaze durations per zone were related to several task and subject variables using analysis of variance. Second, Debic's raw data were reduced to sequences of eye fixations on distinct parts of the problem text.

We will only discuss some main findings of the second analysis. For a report of the results of the first analysis and of the non eye-movement data collected during this study (problem difficulty, typical errors, solution times), we refer to another paper (De Corte & Verschaffel, 1986).

3. Results

The second analysis consisted of two stages. First, graphical representations of the raw eye-movement data were made on millimeter paper. Because we had to make these drawings by hand, the data of only six children have been analyzed until now (three from each ability group). Second, the resulting raw and long diagrams were further reduced by aggregating the measurement data in terms of the following categories.

(1) Sentence reading (S1, S2 or S3) : the child reads the first, the second or the third problem sentence. To be coded in one of these categories, the child's eye-movements must show the typical eye-movement pattern of reading behavior, i.e. subsequent fixations in the distinct zones of that particular sentence from left to right.

(2) Number reading (N1, N2) : the child is looking at the first or the second given number in the problem.

(3) Word reading (W1, W2, W3) : one or more words in the first, the second or the third sentence respectively are viewed. A piece of an eye-movement diagram was scored in one of these categories, when it contained fixations that could not be conceived as sentence or number reading.

The results of this data reduction were again graphically represented, using whole boxes, half boxes and small lines referring to whole sentence, word and number reading respectively. The horizontal location of the boxes and lines refers to the distinct sentences, words and numbers. The length of the boxes and the lines indicates the duration of that particular category; in this case every millimeter represents 200 milliseconds. As an illustration we first present two of these diagrams. Afterwards some more general findings of our analysis of these reduced eye-movement diagrams will be presented.

Figure 1 shows the eye-movement diagrams of Joelle on the Change 1 ("Pete had 5 apples; Ann gave Pete 8 more apples; how many apples does Pete have now ?") and the Change 3 problem ("First Pete had 5 apples; now Pete has 12 apples; how many apples did Pete get more ?". This girl from the L-group solved both problems very quickly : in 16 and 14 seconds respectively. While the former was solved correctly, the latter was answered with a WO error, i.e. an addition instead of a subtraction with the two given numbers. Interestingly, the eye-movement patterns were very similar for both problems : first there was typical reading behavior, involving, however, only the first and the second sentence. Afterwards Joelle's eyes jumped immediately toward the two given numbers, suggesting that she was "doing something" with them. From her answers to all problems we know that Joelle each time added both numbers. For the Change 1 and the Change 3 problem this strategy yielded the correct answer and a WO error respectively.

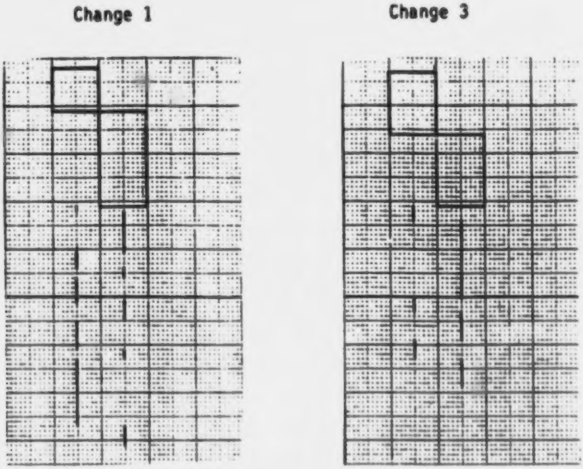

Figure 5 Joëlle's reduced eye-movement protocols for the Change 1 and the Change 3 problem

We examined whether the eleven word problems were completely read by the six children involved in our analysis. In 15 of the 66 cases - almost 25 % - the problem was not fully read; each time the child neglected to look at the question. Eleven of these incomplete readings were coming from one child, namely Joelle (see Figure 1); the other four cases were produced by two other children. Based on the analysis of the gaze durations, we expect the observed frequency to be representative of all twenty children in this study. Elsewhere, we have argued that theoretically, very superficial as well as deep-level processing strategies may account for incomplete reading (De Corte & Verschaffel, 1986). The regularity in Joelle's eye-movement protocols on the distinct problems, together with her error and solution time data, point toward the first hypothetical explanation. For the other cases further probing questions and the presentation of more problems would have been necessary to exclude one of the two alternative interpretations.

We also analyzed what happened during and after the initial reading of the problem text. These data suggest some interesting hypotheses that need, of course, verification using the data of the other children in our study.

First, there seems to be a relationship between problem difficulty and children's eye-movement patterns. The two easiest problems - Combine 1 and Change 1 - were initially read very smoothly, and once the first reading was finished there was almost no rereading of the words and sentences in the problem : the children almost exclusively looked at the numbers. The more difficult problems - Change 5 and Compare 5 - on the other hand elicited a lot of rereading of numbers, words and whole sentences during the initial reading of the problem. But also after having read them for the first time, these difficult problems continued to elicit more fixations on the words and whole sentences than the easy ones. Taking both findings together, this suggests that, especially when children are confronted with a real problem the solution process does not occur as a linear sequence of sharply distinguished stages, namely a representational and a computational stage. On the contrary, both aspects seem to alternate and interact in real problem solving.

Second, the eye-movement data also suggest a relationship between children's problem-solving ability on the one hand and their reading pattern on the other. We found not only that most rereadings during the initial reading of the problem text came from the children of the H-group, but there was also a qualitative difference between both groups. While rereading of the L-children consisted almost exclusively in jumping back and forth to the numbers in the problem, the children from the H-group frequently reviewed words and even whole sentences too. Moreover, after the first reading of the problem, non-numerical aspects (words and sentences) were reviewed more frequently by the children in the H-group than by the L-group children. Relating this to the hypothesis specified in the preceding paragraph, it seems plausible to assume that high-ability children have a more extensive representational stage consisting of semantic processing than their low-ability counterparts.

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A STUDY OF CHILDREN'S MATHEMATICAL PROBLEM-SOLVING HEURISTICS

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To teach problem solving in mathematics, it is important to understand the competencies contributing to "problem solving ability"--how they can best be characterized in detail, how they develop in children, and how they are organized into broader cognitive systems. In studying planning and executive control in individual problem solvers, the heuristic process is often taken as a unit for analysis. Thus we need a psychologically sound theory of competencies associated with heuristic processes which includes their measurement, their developmental sequences, and their pedagogy. This paper reports observations made during the first part of a study of children's heuristic process usage.

The study as a whole envisions the investigation of three major heuristic plans: "think of a simpler problem" (TSP), "trial and error" (TE), and "special cases" (SC). The design of a script for conducting structured clinical problem-solving interviews with children based on TSP has been described elsewhere (Goldin, 1985a). The script itself, 19 pages in length, is also available (Goldin, 1985b). Organization of the script is based in part on earlier work by Goldin and Germain (1983). Following the script nearly verbatim, the clinician guides the subject through a complex plan for solving the problem, "What is the remainder when two to the 50th power is divided by three?" The major script sections are: (I) Explanation of prerequisites for understanding the problem; (II) Presentation of the main problem; (III) Guided use of the heuristic process TSP, without "correcting" any prior conceptions or misconceptions; (IV) Presentation of simpler problems in sequence, as appropriate; (V) Guided detection of the pattern in the sequence of remainders; (VI) Guided conjectured solution to the main problem; (VII) Depth of understanding; (VIII) Looking back. The script provides for various response alternatives. Whenever the clinician asks a question or makes a suggestion, the child is permitted to work freely until a conclusion or impasse is reached. The next question or suggestion follows without correcting the previous work. For each subprocess encountered, the intent is to assess competence: (i) Is the subprocess used spontaneously, or when the child is prompted, or not at all? (ii) Is the child's spontaneous or prompted use of the subprocess successful?

In the fall of 1984, pilot interviews with academically talented children in grades 4-6 were conducted during the development of the script. In an earlier paper (Goldin and Landis, 1985a), excerpts from one such interview are discussed and the proposed scoring system described. The complete transcript of this child's interview is also available (Goldin and Landis, 1985b). In the spring of 1985, after completion of training sessions for the clinicians, 28 children were interviewed. Subjects were grade 4-6 students enrolled in a Saturday morning program for academically talented children at Montclair State College in New Jersey. As this is written transcripts have been completed for 22 tapes, and scoring is in progress. The goal is to create a profile of competencies for each subject, as well as a composite profile describing the development of subprocesses.

To illustrate, Table 1 shows the competency scoring for Linda (L), age 9, grade 4, and James (J), age 11, grade 5 (not their real names). Neither had seen exponents before. The following are some highlights from the interviews.

L proposes (2 x 50) ÷ 3 as a way to solve the main problem, and seems satisfied. When asked to think of a simpler related problem, she first tries to simplify the division process, then proposes $4^4 \div 4$, and next $2^{30} \div 3$, R = (when prompted). She calculates (2 x 30) ÷ 3. After being presented with $2^2 \div 3$, R = , she is able when prompted to think of $2^5 \div 3$, R = and $2^6 \div 3$, R = . Having found that $2^5 = 32$, she realizes, "two fifty times ... well that would be a hundred ... no ... wait a second ... here ... giggle, giggle ... wait ... okay ... two fifty times wouldn't be a hundred because ... I think it would be too ... umm ... that would be a hard problem to solve ..." She continues, "... two fifty times would be a lot. I would think. Because even ... two times five [referring to $2^5 = 32$] is a lot." When presented with $2^7 \div 3$, R = , L spontaneously states, "... if you get, if you do two six times you would get a 64, and the next time if you wanted to do seven times, all you would have to do is times it by two one more time." She spontaneously finds a pattern in the reviewed sequence of problems, and extends it to the original problem: "... if you keep on going one two one two for like 50 times, then you would get to one ..." and "... well it seems here like if it's an even number on the top, you get the odd remainder ..." and "... so it would seem that when you got to 50 the remainder of 50 would be umm ... well 50 is an even number so it would be one."

J also calculates (2 x 50) ÷ 3. When asked to think of a simpler related problem, he suggests $2^{40} \div 3$ because 80 ÷ 3 would be easier than 100 ÷ 3. He does not think this would help with the original problem, "... because you may not get

Table 1. Competency Scoring for Linda (L), Subject #6, and James (J), Subject #25.

Section of script and process or subprocess in which competence is scored	Uses indicated process . . . not at all		when prompted		spontaneously		successfully	
I. Explanation of prerequisites for understanding the problem	L	J	L	J	L	J	L	J
a. explains meaning of 3^4	✓	✓						
b. calculates value of 3^4 first trial (1)			✓	✓			no	✓
second trial (2)		*	✓	*		*	✓	*
c. calculates value of 5^3					✓	✓	✓	✓
d. calculates value of R for 17 ÷ 5, R = ...					✓	✓	✓	✓
e. explains meaning of 17 ÷ 5, R = ...			✓	✓			✓	✓
f. calculates 3^2 ÷ 5, R = first trial (1)			✓			✓	✓	no
calculates 3^2 (2)	*		*		*	✓	*	no
explains meaning of 3^2 (3)	*		*	✓	*		*	no
calculates 3^3 (4)	*		*		*	✓	*	✓
calculates 4^3 (5)	*		*		*	✓	*	no
explains meaning of 4^3 (6)	*		*	✓	*		*	✓
calculates 4^3 (7)	*		*		*	✓	*	✓
calculates 3^2 (8)	*		*		*	✓	*	no
calculates 3^2 (9)	*		*	✓	*		*	✓
calculates 9 ÷ 5, R = (10)	*		*		*	✓	*	no
calculates 9 ÷ 5, R = (11)	*		*	✓	*		*	✓
g. calculates 5^2 ÷ 6, R = ...	*		*		*	✓	*	✓
h. explains meaning of 5^2 ÷ 6, R = ...	*		*	✓	*		*	✓
i. calculates 6^2 ÷ 5, R = ...	*		*		*	✓	*	✓
j. explains meaning of 6^2 ÷ 5, R = ...	*		*	✓	*		*	✓
II. Presentation of the main problem	L	J	L	J	L	J	L	J
a. employs a simpler method or rule					✓	✓	no	no
b. calculates (2 x 50) ÷ 3, R = ...					✓	✓	✓	✓
c. monitors for correctness of method or rule	✓	✓						
d. decides to think of a simpler related problem ..	✓	✓						

* = not applicable

Table 1 (continued).

	not at all		when prompted		spontaneously		successfully	
III. Guided use of the heuristic process TSP	L	J	L	J	L	J	L	J
a. decides to think of a simpler related problem								
first trial (1)			✓	✓			no	no
second trial (2)			✓	✓			no	✓
third trial (3)			✓	✓			✓	?
b. monitors for relatedness to original problem ..	✓	✓						
c. solves simpler related problem		✓			✓		no	
d. generates sequence of related problems	✓	✓						
e. conjectures solution to original problem	✓	✓						
IV. Presentation of simpler problems	L	J	L	J	L	J	L	J
a. calculates $2^2 \div 3$, R = (presented)					✓	✓	?	✓
b. draws inferences for original problem			✓	✓			no	no
c. decides to think of another simpler problem			✓	✓			✓	✓
d. solves simpler related problem								
first trial (1)				✓	✓		no	no
second trial (2)			✓	✓			✓	no
e. draws inferences for original problem		✓	✓				✓	
f. decides to think of another simpler problem			✓	✓			✓	no
g. solves simpler related problem		*	✓	*		*	✓	*
h. draws inferences for original problem		*	✓	*		*	✓	*
i. calculates $2^3 \div 3$, R = (presented)	*		*		*	✓	*	✓
j. draws inferences for original problem	*	✓	*		*		*	
k. decides to think of another simpler problem	*		*	✓	*		*	✓
l. solves simpler related problem	*		*		*	✓	*	no
m. draws inferences for original problem	*	✓	*		*		*	
n. calculates $2^4 \div 3$, R = (presented)	*		*		*	✓	*	✓
o. draws inferences for original problem	*		*	✓	*		*	✓
p. conjectures solution to original problem	*		*		*	✓	*	no
q. monitors for correctness of conjecture	*	✓	*		*		*	
r. decides to think of another simpler problem	*		*	✓	*		*	✓

Table 1 (continued).	not at all		when prompted		spontaneously		successfully	
s. solves simpler related problem	*		*		*	✓	*	no
t. draws inferences for original problem	*		*		*	✓	*	no
u. calculates $2^5 \div 3$, R = (presented)	*		*		*	✓	*	✓
v. draws inferences for original problem	*		*	✓	*		*	no
w. calculates $2^6 \div 3$, R = (presented)								
first trial (1)	*		*		*	✓	*	no
second trial (2)	*		*	✓	*		*	✓
x. draws inferences for original problem	*	✓	*		*		*	
y. calculates $2^7 \div 3$, R = (presented)			✓			✓	✓	✓
z. draws inferences for original problem		*		*	✓	*	✓	*
a. conjectures solution to original problem	✓	*		*		*		*
b. generates sequence of related problems	✓	✓						
c. looks for pattern in the sequence of problems	✓	✓						
V. Guided detection of pattern in remainders	L	J	L	J	L	J	L	J
a. finds pattern in reviewed sequence of problems					✓	✓	✓	✓
b. monitors for correctness of pattern		✓			✓		✓	
c. conjectures solution to original problem based on pattern				✓		✓	✓	✓
d. recognizes conjectured solution as conjecture, seeks reason behind pattern	✓	✓						
VI. Guided solution of the original problem	*	*	*	*	*	*	*	*
VII. Depth of understanding	L	J	L	J	L	J	L	J
a. applies pattern to $2^{44} \div 3$, R =					✓	✓	✓	✓
b. applies pattern to $2^{75} \div 3$, R =					✓	✓	✓	✓
c. recognizes conjectured solution as conjecture		✓			?			
d. describes application of TSP to $3^{50} \div 4$, R =		✓			✓		✓	
e. seeks reason behind pattern	✓	✓						

Table 1 (continued).

	not at all		when prompted		spontaneously		successfully	
VIII. Looking back	L	J	L	J	L	J	L	J
a. identifies related problem		✓	✓				✓	
b. identifies reason for relationship as TSP or pattern identification	✓	*		*		*		*
c. expresses interest in looking back and does so .	✓			✓				✓
d. gives coherent retrospective account	*		*	✓	*		*	no
e. corrects own conceptual misunderstanding(s)	*		*		*	✓	*	✓

the same remainder and you probably won't get the same answer." After being presented with $2^2 \div 3$, R = , $2^3 \div 3$, R = , and $2^4 \div 3$, R = , J observes, "I tried two times 50, but I guess that's not the same, so what I have to do is 50 times 50 ..." He also realizes that 2^7 is 2^6 times 2. He spontaneously and successfully detects the pattern in the reviewed sequence of problems: "If it's an even number and you put divided by three the remainder is one; if it's the odd number it's 2."

In both interviews what seem to be chaotic, disorganized problem-solving processes, based on major misconceptions, rapidly change into organized processes in which patterns are detected and solutions to the original problem found. Certain processes (such as the pattern recognition and extrapolation) occur spontaneously and are used competently; while others (such as deciding to think of simpler related problems) occur only when prompted and with but intermittent success. We hope that such observations, if generalizable across a wider population, will provide the beginning of a developmental theory of heuristic processes in children.

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'PROCEDURAL THINKING' BY CHILDREN AGED 8 - 12 USING TURTLE-GEOMETRY

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INTRODUCTION

One of the attractive features of LOGO is the ease by which its basic vocabulary can be extended. Thus, any LOGO production of a geometric or numeric object can be turned into a procedure simply by naming it (with the prefix TO ...). The name becomes a new LOGO word and the production can be evoked simply by typing the name. In this paper, we shall restrict ourselves to Turtle Geometry and to the production of geometric figures.

We can distinguish three uses of procedures:

i) as a mean of saving a production of a figure so as to be able to reproduce the figure on the computer screen

ii) as a mean of editing (e.g. debugging,modifying,generalizing) a production

iii) as a programming technique with its available procedural mechanisms such as calling subprocedures, iteration, variable and recursion (Abelson and diSessa,1981)

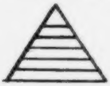
We shall focus mainly on the latter use of procedures and, in particular, on the use of subprocedures in the productions of complex figures.

MATHEMATICAL-PRORAMMING LINKAGES

Turtle Geometry can be considered, first of all, as a particular type of geometry with its underlying concepts, methods and characterization of geometric shapes. Its abvious mathematical-geometrical nature is actually independent of the use of computers. However, the fact that it is embedded within a computer language not only provides for a very different way of "doing mathematics" but also brings into play some interesting links between programming concepts and geometric ones. Among these we include relations such as those of:

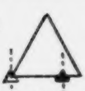
procedure to figure: Figures on the computer screen have particular position and orientation viz-a-viz the (implicit) centre and the vertical-horizontal axes. On the other hand, due to the nature of Turtle Geometry, productions of figures are intrinsic descriptions and hence the invariants of a procedure is a class of congruent screen figures (or, possibly a larger class if variables are used in the procedure). It is this relation that allows for the easy use of a procedure in the production of complex patterns of translated and rotated figures.

figure to procedure: Even when using only the commands FD, BK, RT, LT (hence viewing a figure as being composed of line segments only), there are many possible productions of a figure. These may vary greatly to the extent in which they reflect inherent mathematical properties of the figure they describe (properties such as symmetry, proportionality relations among different lengths, supplementarity of angles, etc.). But the availability of procedural mechanisms suggests the possibility of quite different perceptual organisations of a figure. For example, the figure

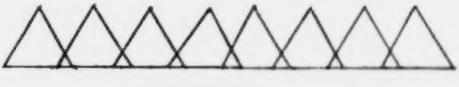


initially percieved as 8 line segments, may now be decoposed in terms of a triangle and 6 line segments (i.e. calling a single procedure) or 6 embedded triangles (i.e. calling a procedure with a variable). More dynamically, it may be thought of as an iterative construct (using REPEAT and incrementing the variable) or a recursive one. Thus, different programming ideas provide for different conceptions of the structure of geometric figures.

subprocedure to subfigure: When a figure is identified as a component of another figure, its procedure now becomes a subprocedure in a new production. Such use of procedures brings into play the notions of turtle state and interface, i.e. a procedure now has to be understood in terms of changes in 'turtle state' and not simply identified with its output on the screen.

Some aspects of 'state' are, in a sense, programming artifacts which do not always relate to any mathematical property of the figure involved. For example, in a procedure TO TRIANGLE the initial and final 'turtle states' might be arranged as follows  (Δ = initial state, ▲ = final state)

because one was intending to use it as a subprocedure in the production of



Clearly, though TRIANGLE will produce a triangle on the screen, it is in no way a 'canonical' intrinsic description of a triangle since it incorporates some extraneous interface component. It's utility lies in its use as a subprocedure.

There are many situations in which the 'interface problem' bears directly on important mathematical ideas. One example of the fruitful linkage with mathematics is the relation of state transparency to the notion of total turn and the role of 360 (The Total Turn Theorem). Another is the 'angle interface' necessary to produce n-fold rotational symmetry.

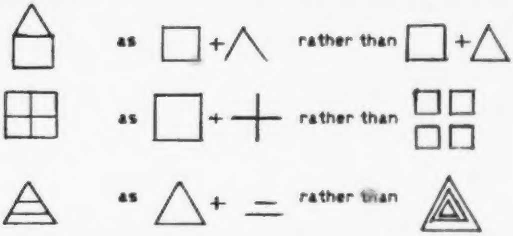
CHILDREN'S TURTLE GEOMETRY; THE 'DRAWING SCHEMA'

Our above analysis of the use of procedures in Turtle Geometry points to some rather subtle notions. We shall try to address the question of the extent to which children come to understand these notions, based on our observations of four children for sixty hours (started in January 1984). However, in order to describe children's 'procedural thinking', it is important to describe their conception of what Turtle Geometry is about. Young children (age 8-12) are often introduced to programming in LOGO through the use of the 'drawing with the turtle' metaphor, i.e. the computer screen is the drawing surface and the turtle is an object to draw with. The success of this metaphor is attested by the evidence of a strong drawing schema underlying the children's choices of goals, productions and planning strategies, as well as, their criterion for success (Mendelsohn,1985; Hillel and Samurçay,1985). Thus, children often set themselves 'concrete' projects of 'drawing' a figure in some specific screen location and orientation and their productions resemble the sequence of actions in paper-and-pencil drawing. Their planning is local in nature and the choices of inputs to commands are based on perceptual cues rather than on inherent mathematical relations. 'More-or-less' solutions can be perfectly acceptable (Hillel et al.,1986)

There are no conflicts within the 'drawing schema' if the child is engaged in 'naive programming mode' (Kieren,1985) in which the use of procedures is mainly for saving and editing. However, we shall give evidence below that such schema is not compatible with 'planned programing mode' and the fluent use of different 'procedural mechanisms'.

perceptual organisation of figures: One of the prequisites for using subprocedures in a production of a figure is to identify a relevant subfigure (whether a procedure for such subfigure exists or is still to be defined). This calls for a particular perceptual organisation of, what is often, a connected figure or one with embedded subfigures. Our evidence points to a strong resonance between the 'drawing schema' and the perceptual organisation of a figure into 'primary contour structures' (Vurpillot,1972). Characteristic of such organisation is the avoidance of overlapping line segments (i.e. no two structures share the same line segment) and the treatment of line segments as undivided units (i.e. no structure contains only a part of a line segment).

The children we have observed, have spontaneously organised figures as follows:

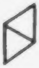


even when they had procedures such as TO SQUARE :X and TO TRIANGLE :X at their disposal.

It appears that a fluent use of subprocedures requires the perceptual organisation of a figure into either 'secondary contour structures' or 'area structures' (see Vurpillot). We are not suggesting here that children of such age are not capable of this kind of organisation, but only that their 'drawing schema' favours a simpler organisation.

awareness of 'turtle-state': When a procedure is used simply as a mean to reproduce a particular 'drawing' on the screen, it can be identified with the figure. In a sense, naming a production (TO...) serves precisely as a way to supress it, by attributing something static (a name) to a dynamic process. It is only when procedures are used as subprocedures that the underlying process becomes important again. Our evidence suggests that children have cognitive difficulties to make the necessary changes to their mental representation of procedures.

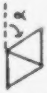
Many research reports on LOGO have documented the persistence of interface bugs by children. However, a careful differentiation has not been made between bugs which are computational in nature and those that stem out a conceptual difficulties in understanding a procedure as more than just the figure. We have already alluded to the fact that figuring out the correct interface may require some mathematical knowledge not yet available to the child. For example, we have given the chidren the task of producing the figure



using TO TRI (REPEAT 3[FD 30 RT 120]). Some of the attempted solutions were:

i) TRI FD 30 RT 45 TRI   ii) TRI TRI   iii) TRI FD 30 TRI

The first attempt clearly indicates awarenss of the relevant 'states'. It failed either because the child could not figure out the angle α in



or because she assumed that it is 45 because "it looks like a 45". On the other hand, attempts ii) and iii) point to, at best, only a partial representation by the child of the actual production of the triangle.

The interface bug exemplified by iii) is the type that we have observed most often. In this case, direct-mode activity provides visual cues as to the final turtle state after a procedure has been executed and there is a partial resolution of the interface. The bugs are often related to the initial state of the next called procedure.

Again, we are suggesting that the 'drawing schema' favours a particularly static conception of procedures. The one discernable change in the chilren's behaviour is their ability, in time, to more quickly identify the source of bugs as being related to interface.

avoidance of the use of procedures: The children's final project (after 55 hours of LOGO experience) was the drawing of an elaborate castle. Their initial preplanned productions, which were quite lengthy, were entirely 'spaghetti style'. They neither broke up the figure into modules nor did they use any of their available procedures to produce some of the abvious parts of their castle such as windows and doors. It would be easy to explain this behaviour simply as a lack of the concept of procedures. Yet, the same children have used, on occasion and unprompted, some very sophisticated procedural mechanisms. We feel that the explanation for this resides partially in the affective domain, and that there are several related issues at stake:

i) 'drawing with the turtle' is a way of being in control, i.e. mentally guiding the turtle around a picture. Calling subprocedures is, in a sense, relinquishing the control to the computer.

ii) experience with subprocedures has alerted the children to the frequent difficulties with interface and subsequent debugging. From their perspective their approach is easier and more efficient (the length of a procedure is not a factor since the production 'disappears' once it is named).

iii) children seem to behave differently with 'picture tasks' such as the castle one mentioned above than with 'abstract tasks' consisting of patterns and designs. Sutherland and Hoyles (1985) have made this observation when looking at the children's use of variables and the same seems to hold for the use of procedures. When a particular figure is repeated often within a design, there appears to be more willingness to abandon the 'drawing schema' in favour of procedures since the 'payoff' is more apparent.

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VALIDATION OF THE HIERARCHICAL STRUCTURE OF A SYSTEM OF SOLUTION STRATEGIES FOR SPEED PROBLEMS

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BACKGROUND AND RESEARCH PROBLEM

Israeli high school students in all grade levels are being taught to solve speed problems algebraically. However, in a previous study (Gorodetsky, Hoz and Vinner, in press) we have found that Israeli high school students have various difficulties in solving speed problems. From the students' solutions a hierarchical system consisting of 6 solution strategies was constructed. The system incorporates two dimensions involved in the solution of speed problems. The *physical representation,* is an elaboration of Paige and Simon's (1966) "physical intuition". It is the semantics of the solution, i.e., the use of the physical referents of the involved concepts, their features and their relationships. Specifically, these concepts are speed, time, and distance, whose relationship can include either two or three concepts. The *mathematical modeling,* which is the procedural mathematical knowledge that is necessary to solve the problem, namely, the mathematical symbols by which the involved concepts, their features and their relationship were represented, and the procedures for their manipulation. The degree of coordination of these two dimensions served to hierarchically order the six solution strategies from the highest ("formal") to the lowest ("confusion"). The hierarchical system of solution strategies is depicted in Figure 1, and a more detailed description is given in Gorodetsky, Hoz and Vinner (in press).

Figure 1. The hierarchy of solution strategies and their description

Solution strategy	Level of physical representation	Level of mathematical modeling	Short description
Formal	Complete	Formal	Comparison of total times of the vehicles, derived by either the correct formulas or other. nonstandard (original) procedures
Intuitive	Complete	Intuitive	Comparison of total times of the vehicles, derived by implicit use of the formula.
Rote	Intuitive	Intuitive	Comparison of "total times" of the vehicles, derived from substituition of incorrect concept values in the formula.
Compensation	Intuitive	Intuitive	Numerical or semi-numerical comparison of "time", "distance" or "speed" gaps between the vehicles which were combinations of the given speed values.
Numerical	Intuitive	None	Comparison of "total speeds", "average speeds", "total distances", or "total times" of the vehicles which were numerical combinations of the given speed values.
Confusion	None	None	Comparison of "total times" which were nonsensical combinations of both given and not given values.

The hierarchical nature of the system of solution strategies was derived by theoretical analysis of the obtained solutions. This presentation is an attempt to validate empirically the hierarchical nature of the system. From the hierarchical nature of the system, two predictions regarding expected shifts of solution strategies were derived : Increased problem difficulty would produce greater use of lower strategies and decreased problem difficulty would produce greater use of higher strategies.

The research hypotheses were that students presented with two problems of different difficulty will exhibit the following behaviors.

1. Students will shift from the strategies used to solve a first more difficult problem to higher strategies when attempting to solve the second easier problem.
2. Students will shift from the strategies used to solve a first easier problem to lower strategies when attempting to solve the second more difficult problem.

METHOD

1. The participants The participants were 563 students from 12 high schools in the Negev (the southern region of Israel). 270 students were from 11 ninth grade classes, 205 were from 10 tenth grade classes, and 88 were from 7 eleventh grade classes. The classes were selected by 18 mathematics teachers in these schools to represent grade levels (9 to 11) and ability levels (average and up). These students were exposed to instruction of algebra word problems (including speed problems) for several periods during their studies.

2. The test The test included the two following questions.

(i) "Two cars start at the same time from city A. They go to city B and return to city A without delay. The distance between the cities is 300 km. Car 1 goes to city B at the speed of 30 km/h and back at 50 km/h, and Car 2 goes both ways at the speed of 40 km/h. Which car returned first to city A?"

(ii) "Two ships start at the same time from port A. They go to port B and return to port A without delay. Ship 1 goes to port B at the speed of 30 km/h and back at 60 km/h, and Ship 2 goes both ways at the speed of 50 km/h. Which ship returned first to port A?"

Both problems are equivalent in every respect except for the presence or absence of the value for the distance. This makes problem (i) easier than problem (ii). Two test forms differing in the presentation order of the problems were constructed. In form A the easier problem (i) was the first and the more

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difficult problem (ii) was second. In form B the more difficult problem (ii) was the first and the easier problem was second. Therefore each test can be viewed as being composed of one problem to which the distance magnitude either was added (in form B) or was deleted from (in form A).

RESULTS

The solution strategies employed by the students in each problem were classified according to the hierarchy of solution strategies. The solution strategies employed in the first and second problems were tabulated in Table 1. The cells of this table depict the persentage of solutions included under every combination of the 6 solution strategies, and two additional solution types (which are not considered strategies). Strategy shifts are idetified by the figures in the off-diagonal cells of Table 1: shifts to lower strategies in the the upper right region and to higher strategies in the lower left region. The diagonal (boxed) cells indicate no strategy shift.

To test the nature of strategic shifts we compared the strategies employed to solve the second problem with those used to solve the first one. The results generally confirm the research hypotheses, as evidenced by the following findings. When problem difficulty decreased (Form B--addition of distance magnitude) it was observed that (1) the majority of shifts were made towards the two correct strategies, and (2) students who employed the correct strategies tended to reemploy them. When problem difficulty increased (Form A--deletion of distance magnitude) it was observed that (1) most students reemployed their incorrect strategies, and (2) relatively few strategic shifts were made, most of them towards lower strategies (including shifts from correct strategies to producing no solution or to using only partial data).

Table 1. Strategy shifts following an increase and decrease in problem difficulty

Strategy in Problem 2	Formal			Intuitive			Rote			Compensation			Numerical			Confusion			NS			Partial		
Grade level	9	10	11	9	10	11	9	10	11	9	10	11	9	10	11	9	10	11	9	10	11	9	10	11
Strategy in Problem 1	Addition of distance magnitude (Form B)																							
Formal	100	92	67				0	0	33	0	8	0												
Intuitive	30	20	0	60	20	87				0	30	0	10	20	13				0	10	0			
Rote	75	56	25	25	0	0	0	33	75										0	11	0			
Compensation	13	13	29	29	0	0				52	74	57	0	0	14				6	13	0			
Numerical	2	12	16	32	18	21	0	3	5	6	3	0	53	62	42	4	0	5	2	3	11			
Confusion	0	40	0	0	20	0				0	20	0				100	0	100	0	20	0			
Ns	11	18	17	44	55	66	6	0	0	6	0	0							33	27	17			
Part	25	50	0	55	12	100	0	12	0	12	12	0							12	12	0			
	Deletion of distance magnitude (Form A)																							
Formal	50	54	63	4	7	6	18	11	0	4	4	6	0	11	6				25	14	19	29	14	19
Intuitive	5	15	6	37	25	50				5	10	6	29	20	12	3	0	0	21	30	25	18	35	31
Rote							33	50	100	0	25	0	17	0	0	17	0	0	33	25	0	17	25	0
Compensation										100	71	0	0	14	0	0	14	0				0	14	0
Numerical	5	0	10							14	0	0	81	88	80				0	12	10			
Confusion										0	100	0	33	0	0	67	0	0						
Ns										14	0	0	14	0	0	14	0	0	58	100	100			

Note: 1. Percetages were computed within each row (strategy in first problem); the total frequency in each row included unclassifiable solutions. 2. Ns = no solution was given. 3. Part = solution used only partial data. 4. No partial solutions were given in problem 1 of Form A.

The theoretical analysis was empirically validated by our findings. It gains additional support from its similarity to Biggs and Collis' (1982) SOLO taxonomy, which was based on Piaget's theory.

The hierarchy of solution strategies may be used to add new perspectives to the qualitative differences that were found in several domains between novice and expert problem solvers (e.g., McKloskey, Caramazza and Green, 1980; Chi, Glaser and Rees, 1982).

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Consistency of Strategy Usage in Structurally Equivalent Problems

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The purpose of the present study is fourfold. First, it attempts to identify and classify strategies used in solving structurally equivalent problems in as much as these strategies are indentifiable from the written solutions of the students. Secondly, it investigates the extent to which consistency in strategy usage is exhibited in solving structurally equivalent problems. Thirdly, it investigates the difference in strategy consistency between structurally equivalent problem in different and similar contexts. Fourthly, it investigates whether reflective intelligence is related to strategy consistency in structurally equivalent problems.

Definitions

After consulting two standard sources (Kilpatrick,1978; Goldin and McClintoch, 1980), the following definitions were formulated and adapted to the specific requirements of the present study:

Structurally equivalent problems. Two problems are structurally equivalent if the algorithms for their solutions are isomorphic.

Strategy. The algorithmic approach followed by a particular student to solve a particular problem.

Context. The embodiment of a particular mathematical problem as well as the language in which the latter is stated.

Consistency. For a pair of problems, two solutions (or two students) are consistent if the same strategy is used in solving each of the two problems.

Reflective intelligence. Reflective intelligence is the functioning of a second order system which: a) can perceive and act on the concepts and operations of sensori-motor system; b) can act on them in ways which take account of these relationships and of other information from memory and from the external environment; and, c) can perceive relationships between these concepts and operations (Skemp, 1961, p. 49).

Procedure

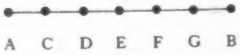
Sample

A sample of 120 students from grades six (11-year old), seven (12-year old), nine (14-year old), and 10 (15-year old) was drawn from three private schools (two for girls and one coeducational) serving a predominantly middle class community in Beirut. Twelve responses were nonusable and consequently were eliminated from the sample of the 108 students, 83 were girls and 25 were boys.

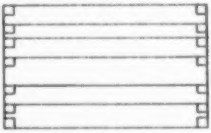
Problems

Two problems T_1 and T_2, each in two contextual settings (mathematical and real world), were constructed and written in English which is a second language for the subjects and the language of instruction for mathematics in their schools. Two equivalent problems were constructed for each of T_1 and T_2. The problems were the following:

A_{1M} How many line segments do you have in the figure?



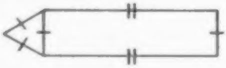
T_{1M} How many rectangles are there in this figure?



T_{1R} In a district there are 7 towns. There is a road between each two towns. How many such roads are in the district?

A_{2R} There is a total of 18 girls and boys. Teams consisting of 2 girls and 4 boys are to be formed. How many of the 18 should be girls and how many should be boys to form the largest number of separate teams?

T_{2M} The figure shown is made up of 4 small segments and two large segments. We have a total of 18 segments of both. How many of the 18 should be small and how many should be large to construct the largest number of figures like the given one?



T_{2R} The total number of tables and chairs is 18. Each table can take exactly 4 chairs. How many of the 18 should be chairs and how many should be tables to form the largest number of tables with 4 chairs each?

The numerical answers for A_{1M} and A_{2R} were given. For all the problems, the instructions to student were to solve the problem and write all their work. In addition each student was given Parts I and II of Skemp Test (Skemp, 1961). The students were divided on the basis of the total score into an upper and lower groups.

Analysis and Results

The written solution of each version of the two problems was analyzed in an attempt to identify the strategy used in solving the problem. In many cases interpretation and /or inference have to be used. The strategies which were identified for T_1 were: (a) Writing and solving a mathematical sentence; (b) non-combinatory counting; (c) unsystematic combinatory counting; and (d) systematic combinatory counting. For T_2 the strategies were: (a) focussing on one variable; (b) approximation - verification; and, (d) construction - verification

The distributions of strategies used in solving each of the versions of T_1 and T_2 were constructed. The percentage of unidentifiable strategies ranged from 9.2% to 20.4% for T_1 and 17.9% to 20.8% for T_2. It is felt that with more strict and specific test instructions the percentage of unidentifiable strategies could be reduced further.

The proportion of consistent students for each problem pair was compared with a proportion of 0.5. All the z-values were not significant ($p < .05$) with the exception of the pair (T_{2R}, T_{2M}) for which the proportion of consistent students was significantly higher than the proportion of nonconsistent students. Consequently it seems that with the exception of (T_{2R}, T_{2M}) students were as likely to be consistent as not consistent in solving structurally equivalent problems.

To investigate the relationship of context to consistency, the consistency variable for each pair of problems having different contexts was cross-tabulated with the consistency variable for the pair of structurally equivalent problems having the same context. The z-test for correlated proportions was used to compare statistically the proportion of consistent students in one pair of problems having different contexts to the porportion of consistent students in one pair of structurally equivalent problems having the same context. Only two z-values were significant ($p < .05$) indicating differences opposite to the expected direction. Thus, the results do not provide evidence to support the hypothesis that students tend to be more consistent in strategy usage when they solve structurally equivalent problems with the same context than with different contexts.

To investigate the relationship between consistency and reflective intelligence, the two variables were cross-tabulated for each problem. χ^2 was significant ($p < .01$) only for the two pairs (A_{1M}, T_{1M}) and (A_{2R}, T_{2R}). It seems that the upper reflective intelligence group used more often consistent strategies than the lower reflective intelligence only for pairs of problems, which have the same context and for which the goal was well-defined i.e. the numerical answer was given.

Discussion

The expected high degree of consistency in strategy usage in solving structurally equivalent problems was not substantiated. There was even partial evidence against the expectation that consistency in strategy usage was higher for structurally equivalent problems having the same context than for structurally equivalent problems having different contexts. Two rival hypotheses may be presented to explain these results. One is that the expected consistency was confounded by the way the variables were defined. In particular, the context variable was defined to include both the embodiment of the problem and the language in which the latter is stated. The confounding might have masked the consistency in strategy usage for problems T_{2R} and A_{2R} which are structurally equivalent and have the same context according to the definition; but they are syntactically different. However, this hypothesis does not explain the lack of consistency in strategy usage for problems A_{1M} and T_{1M} which are structurally, contextually, and syntactically equivalent. The second hypothesis is that structural equivalence is not sufficient for consistency in strategy usage and that the critical factor in this respect is the information processing demands of the task. The second hypothesis may explain the lack of consistency for problems A_{1M} and T_{1M} where T_{1M} is judged to demand processing more chunks of information simultaneously than A_{1M}. The second hypothesis would explain the discrepancy in the same way that the information-processing theories explain the decalage in Piagetian tasks (Baylor et al. 1973). Further research is needed to test both hypotheses.

The transfer of strategy for the upper reflective intelligence group, although limited, suggests its potential as a cognitive measure in problem solving tasks which require high-order processing.

At last, the use of written solutions to identify strategies should not be dismissed. There is room for improving its potential not only as a research tool but as a diagnostic tool in instruction.

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STUDENTS' STRATEGIES AND REASONING PROCESSES IN COMPUTER EDUCATIONAL GAMES

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Children usually enjoy all kinds of games and tend to view microcomputers as friendly gaming machines (Greenfield, 1984). Traditionally, games were not regarded as suitable educational activities. However, a gradual change in this respect occurred in the last decades, mainly as a consequence of the emergence of new societal values.

A number of quite interesting and challenging educational games have been developed. It is necessary to assess the educational value of these programs and to consider their cognitive, affective, and social implications (Ponte, 1986).

The most important feature in a game is the existence of a defined goal, which is opposed in a systematic or random way by one or more adversaries, according to some well defined set of rules. To be considered educational, a game must be able to make a specific or general contribution to the process of children's growth, either in terms of learning, motivation, or development of self-confidence.

This study is concerned with the use of concepts by children and their thinking strategies playing educational games. It used four computer games, all dealing with number concepts (factor, prime, negative number, order relation, and approximation) but requiring distinct strategies. These games were either developed or adapted at the Departamento de Educação da Faculdade de Ciências da Universidade de Lisboa.

Pilot work was conducted with 12 elementary and middle school children yielding minor modifications in the games and suggesting strategies of data collection. In the formal study, subjects were 16 fifth- and sixth-graders at a school in a suburban area, not far way from Lisbon. Most of the students had worked with computers before, either at home or in school extra-curricular activities.

Every game was played for about 30 minutes. The students worked in pairs or in groups of three, allowing the researchers to follow their discussion as they went through each game. In one game, data was also collected by the computer, recording students' critical moves.

The games were played in the order in which the main observations and results are reported below. The researchers introduced each game with a brief demonstration, explaining its rules, but keeping themselves of giving any clues concerning possible winning strategies, except in a few cases that will be specifically commented.

Observations and Results

WHERE IS TRAQUINAS HIDDEN? This game is a modification of the well-known HURKLE. The character, Traquinas, jumps around a few places and finally hiddens in a number line marked from -10 to +10. The student is asked to guess the place were Traquinas is hidden and following each erroneous guess a clue is given (larger number, smaller number). Wrong guesses yield scores that increase quadractically with the number of attempts.

Since students had not been taught about negative numbers before and some of them were not completely familiar with the computer keyboard, there was a brief informal introduction on how to get these numbers on the computer.

Observations:

(1) All students understood the gaming situation: Traquinas is hidden in one and just one place. They readily accepted that negative numbers were some sort of an "extension" of the natural numbers. Some of them said to know these numbers, mostly in connection with temperatures. Others did not immediately understood that each mark in the number line corresponded to one number. However, this kind of difficulty seemed to be easily overcome. Scores did not attract the attention of the students.

(2) There were marked differences in the facility with which students grasped how to extend the relationship "greater than" to the new number context. Some of them thought that the number immediately left to 0 was -9. After a few trials some students appeared to have understood pretty well this relationship, while even in the end of the game others seemed still quite confused.

Some students interpreted the clue, "it is in a larger number" as "it is in a positive number," and the clue "it is in a smaller number" as "it is in a negative number." This is a good example of a tendency to think about relative entities in absolute terms. In these cases they were helped to clarify the notion of larger and smaller, to be able to properly interpret the clues given by the computer. In most instances these explanations were understood, at least as could be observed in subsequent trials of the game.

(3) A number of strategies were identified:

(a) "Jump around." If the computer says that the number is larger, jump to a much larger number. In most cases only the information conveyed by the last trial was taken into account, leading some students to guess for more than once in the same number.

(b) "Go up or down just one number." This is a "safety strategy," which in general yields high scores.

(c) "Systematic division." Start with zero. Then proceed dividing in two equal parts the intervals were Traquinas is said to be.

(d) Search for a "smart strategy." He may be hidden in the last place he showed up..., or, perhaps, in the place where he glanced for the first time...

The game was useful in providing a context for the introduction of the number line representation of integer numbers. Even when the concepts were not immediately grasped, a brief discussion seemed to provide enough basis for understanding.

MULTIPLICATION CONTEST. This game is a competition among two to four players, requiring the execution of single or multidigit multiplications. In each question each player is allowed at most three guesses. The score is a function of the number of guesses and the time required to give the correct answer. The player that first reaches a predetermined score wins the game.

Students were suggested to play in a way such that all multiplication questions involved numbers below 15. This game induced an easy involvement of all the children and its general features and purpose were immediately understood.

Observations:

(1) Some students consistently failed in some multiplication facts (6x7, 9x7, 6x8, ...). Several students never used fast strategies of multiplication by 10. Again, some students did not pay much attention to the time counting features of the game.

(2) At least in some trials all students used some mental representation of the multiplication algorithm. Some students accompanied this representation with the figuration of the algorithm using their fingers. To several students this mental representation was not effective as they failed consistently to multiply by the second figure ("twelve times three,... six, carry one,... sixteen"; "twelve

times nine... two times nine is eighteen, carry one,... twenty eight").

(3) Some students used the commutative property to remember some multiplication facts ("five times six... six times five is thirty"). Only two students were noted using distributivity to remember a multiplication fact ("seven times five... six times five is thirty... thirty five").

With an encouraging presence of the teacher, this game appeared to provide an enjoyable setting to recall some basic arithmetic and to practice mental computation.

BIG ESTIMATION CONTEST. In this game each player is asked an estimation of a multidigit multiplication. As the previous game, it is a competition among two to four players, with three guesses to each question. The score is a function of the number of guesses, the accuracy of the response, and the time required to give an acceptable estimation.

The game was introduced with a brief demonstration in which students were explained the concept of approximation.

Observations:

(1) Some students refused completely the concept of approximation, and played the game as if it required precise responses. Others, although seemingly understanding the general idea, had no tolerance for errors and preferred to give exact answers ("more or less is not good"), spending a lot of time in each question.

(2) Several strategies were noted:

(a) Rounding. Just round one of the numbers to the nearest tenth. Example: 19x13 --> 20x13.

(b) Double rounding. Round both numbers to their nearest tenths. Example: 19x12 --> 20x10.

(c) Rounding with compensation. Use one of the above strategies and add or subtract a compensanting quantity. Ex. 97x72 --> 100x72 - something.

(3) Rounding and double rounding strategies were explained in the beginning and were most commonly used.

(4) For some students the idea of rounding was difficult to grasp and they preferred to truncate the numbers. Example: 17x12 --> 10x12.

(5) Most students had no idea of the size of the numbers they were going to obtain. For example, the double rounding strategy led one student to reason that: 4x39 --> 1x40, and 2x17 --> 17.

This game was acceptably understood by part of the students and poorly by others. It required a big conceptual leap from students' previous experience. To be useful in the classroom the game needs a lot of teacher involvement and support and the articulation with other estimation oriented activities.

TRINCA-ESPINHAS. This game is a Portuguese adaptation of the popular TAXMAN. From a given list of numbers we pick up numbers and the computer picks its divisors. Only numbers with divisors on the list may be picked by us and in the end the numbers left are taken by the computer.

The game was introduced with a demonstration trial, using a list of 12 numbers. The first number picked was 10, and that was used to explain the rules. Students were then encouraged to play by themselves in a few trials with lists of 12 numbers. If they did not succeed in winning the computer, they were further suggested to come down to lists of 7 or even of 5 numbers. From a winning point, students were then encouraged to play with larger lists of numbers.

Observations:

(1) Almost all students succeeded in understanding the game. The fifth year students who had not been formally introduced to the mathematical notions of divisor and multiple, involved in the game, showed easiness in using these concepts.

(2) Strategies:

(a) What is the first number to take? After several trials most students realized that it was the largest prime.

(b) What are the next numbers to take? Some students tried numbers with just one factor, beginning in the smaller numbers on the list. This is a sort of a "Safety strategy" leading to poor results. Some others seemed to follow a similar strategy, but beginning in the higher numbers on the list. This strategy was not always strictly applied, probably because some of the possible candidates were overlooked, but generally conducted to good results.

(3) The largest number may conduct to two almost opposite situations. Either the desire of taking it immediately, regardless of its divisors, and some students took first 12 in the 12 number list, or the desire of not taking it at all, and some students disregarded it in the 7 number list. Also, numbers with many divisors (like 12, 8, and 6) are a temptation for an early collection, almost always with very poor results.

This was the most enjoyed game, despite the fact that no student was able to get better than a win with a list of 25 numbers. Fun seems to be higher when children perceive a real challenge from the computer but feel able to overcome it.

Conclusions

Overall the students enjoyed the two-hour session in which they played these four games. They were becoming somehow tired with the sequence multiplication-estimation, but wellcame the last game, TRINCA. To end the sessions it was necessary to declare them over, since students would not leave just by themselves.

The requirement of mathematical concepts not previously studied was not a barrier to students' involvement in the games. They provided a stimulating environment for the introduction of these concepts, on which formal teaching could build. This supports Bright et al. (1985) contention that games can be used in pre-instructional settings. The first contact of young children with computers induces usually lots of excitement (Malone, 1982). This early enthusiasm does not stand for a long time, but can be used to foster an initial positive contact with modern technology.

Besides providing a stimulating learning environment (Kraus, 1984), educational games also allow teachers to obtain a more global view of their students' processes and difficulties. However, one should be remainded that the use of computer educational games should always be well planned. To be of real educational value, games should be components of a more general set of activities to be performed in articulation with them.

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THE 'LEAST NUMBER WINS' GAME ON A LARGE SAMPLE

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The readers of the Hungarian puzzle magazine Füles were invited to take part in the following game. Every player had to choose one integer number and enter it in the entry form cut out from the journal. The winner was the player who sent in the smallest number which has not been chosen by any other player.

The idea of this game stems from the paper of HOFSTADTER (1982) which contains several game ideas in this style. However, this game is much simpler than the games described in Hofstadter's paper. Before the announcement of this game in Füles we had secondary school students gifted in mathematics play this game. The game proved very suitable to make the basic notions of mathematical game theory clear for the students. Nevertheless, the students were not able to adopt anything like an optimal mixed strategy when they were playing. This finding is consistent with the well known results in the psychological literature that people are usually not able to adopt random strategies even if they intend to do so. (See TVERSKY and KAHNEMAN (1974) or DIENER and THOMPSON (1985)).

We have definitely observed that in playing the game in fairly small groups for several rounds the students use different playing strategies. However, the small size of the groups did not allow to identify these strategies. We hoped that by having the game played by a large sample we would have a chance to identify some playing

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strategies, even if the possibility of only one round might bias the results. It was also a great challange for us that we had not been able to give even a rough estimate to some of the most simple questions: how many 1's would be entered, how many percent of the entered numbers would be below 100, etc.

Deeply influenced by the book 'The selfish gene' by DAWKINS (1976), we have supposed that many independent players might quite well approximate an optimal mixed strategy in the game-theoretical sense even if the individual players are not aware of what it is. As we shall see, this hypothesis proved to be true at a fairly good level of approximation. This finding may shed some light on the nature of the difficulties of teaching mathematics: normative, rational aspects may be incorporated in some much deeper strata of behaviour than the apparent, everyday thinking. In the light of the selfish gene theory the possibility arises that this rational distribution of several irrational behaviour strategies might be determined at a genetic level. However, instead of speculating on such vague and far-reaching consequences, let us see the findings of the experiment.

The basic data

There were 8192 entries to the game. Beside the numbers, sex and dwelling place data were available from the addresses of the players. (Inquiring about other data might have spoiled the game). 55% of the players were men. In contrast, usually only about 35% of the solvers of other prize puzzles in this magazine (e.g. crossword puzzles) are men. The dwelling place data roughly agreed with the global statistical expectation.

Several hundred aspects of the distribution of the entered numbers and their digits have been analyzed for sex and dwelling place differences. Surprizingly, no significant difference was found except one. There were 275 players who wrote numbers between 600000 and 650000, 60.2% of them were women. The frequent occurence of this range has a special reason. In the announcement of the game it was pointed out that it might happen that the small numbers all hit each other and so a very large number might be the winner. The number of copies of the Füles was also given as 640000 and it was explained that therefore among the first 640000 numbers there must be a potential winning number. Choosing a number about 640000 may display either a misunderstanding of this message or a false enchoring mechanism, in the sense of TVERSKY and KAHNEMAN (1974). Anyway, this trap has attracted significantly more women then men. 170 numbers were entered over 1000000, but in this range no significant sex difference was found.

A surprizing diversity of numbers was received: 2730 different numbers were entered. There were 2068 numbers that were entered by only one player (25%!). The significance of this diversity will be revisited when the game-theoretical aspects will be analyzed. The most frequent number was 1, entered by 387 players (4.7%). Number 1 was a great challange and it was pointed out in the announcement of the game that this is a sure winning number if you choose it and no one else.

The second most frequent number was the 13, chosen by 164 players (2%). This and some other numbers may display a kind of magic thinking. Furthermore, there were several numbers that sticked out of their neighbours, such as 1111 (37 entries), 333 (17 entries), 1234 (15 entries), etc. Another typical kind of magic thinking may be displayed by the date of year numbers (1900 to 1985). There were 256 such entries. However, all the patterns that could be suspected as the results of magic thinking did not involve more than 8% of the players. (Not all of the 13's were considered as the result of magic thinking, as its neighbours had also several entries: e.g. 12 had 44 entries and 14 had 42 entries. Thus only e.g. for 13 only 121 entries were considered as results of magic thinking.)

66% of the entered numbers were odd numbers. 22% of the numbers ended to 1. It is very surprizing that digit 7 was fairly frequent as a last digit (13%), but it was very rare at all other places: only 2.8% of the non-ending digits were 7. It is possible that people feel digit 7 as a "hidden" digit that does not occur to them, and it really is so. But if they want to avoid the numbers of other people, they like to end up with a 7.

120 turned out to be the winning number. It is interesting, that 119 and 121 had 16 and 15 entries, respectively. People liked to avoid round numbers. All the numbers below 120 were covered at least four times but 94, which had only 2 entries. The first three numbers that have not been entered at all were 165, 180 and 200.

Game-theoretical analysis

The existence of an optimal mixed strategy depends on some model assumptions on the game. However, we have proved, that if an optimal mixed strategy exists, it must consist of strictly decreasing probabilities.

We did not succeed in determining the optimal mixed strategy in an analytic way. A computer simulation was done in the following manner.

We supposed that the number of players is N and the greatest number that may be played is N. (This might bias the original rules of the game, but knowing that the probabilities are strictly decreasing this bias must be very small). If coalition is impossible game theory tells us that the optimal mixed strategy for all the players is the same. The assumption that coalition is impossible is very reasonable in the case of our game.

At the first game all the players choose each number between 1 and N at a probability of 1/N. Suppose now that in this game the winning number is k. At the second game all the players choose all the numbers at a probability of 1/(N+1), except the number k which is chosen at a probability of 2/(N+1). The choice probabilities are changed from game to game in a similar way: if the probability of number m was t/(N+n-1) in the n-th game, then a.) if m is the winning number in the n-th game, then its choosing probability changes to (t+1)/(N+n) for the next game, b.) if m is not a winning number in the n-th game, its choosing probability changes to t/(N+n).

We did not succeed in giving a formal convergence proof of the above procedure, but our experiences were quite favourable: the probability distributions after 10000 games were almost strictly monotonically decreasing for several different N-s.

In the next step the empirical choice frequencies were compared with the theoretical probabilities of the optimal mixed strategy. There were two problems with the comparison: number N of the players in the optimal mixed strategy should have been determined and the empirical distribution should have been smoothed some way. Smoothing was necessary because the digits of the numbers were not symmetrical in meaning (e.g. 22% of the numbers had ended to 1).

The smoothing of the entered number distribution was done in three steps. First, the numbers which may be considered as results of magic thinking were eliminated. Second, the numbers 1 and 600000 to 650000 were also eliminated as results of well tangible thinking strategies of clearly different kind. Third, the numbers were divided into 29 categories as follows: 2--10, 11--20, ..., 91--100, 101--200, ..., 901--1000, 1001--2000, ..., 9001-10000, 10000--.

After this a chi-square test was performed to compare the distribution of the theoretical probabilities with the empirical distribution. In the case of N=15000 and N=20000 the test did not show a significant difference between the theoretical and the empirical distributions at a p 0.1 significance level.

This result seems to justify that the sum of the thinking of many independent persons is quite rational in a normative, game-theoretical sense even if separate individuals may be irrational. Is it possible that the 'selfish gene theory' still works at such a high level of mental performance?

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CHILDREN'S COGNITIVE STRATEGIES IN TWO–SPINNER TASKS

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Abstract. An experiment aimed at the identification of cognitive strategies and auxiliary strategies that are used in specific probability problems was run with 30 children of three age groups: 5 – 6, 9 – 10, and 12 – 14 years. The experimental tasks consisted of roulette tasks with two spinners. Pair comparisons were so constructed that a series of strategies or decision heuristics could be identified. The results show that developmental theories on probability reasoning (e.g. PIAGET, NOELTING, SIEGLER) have to be supplemented if probability reasoning in the context of geometrical probability is conceptualized.

In two–spinner tasks, one has to choose the more favorable roulette disk from two disks with different odds; that means, different proportions of favorable and unfavorable wedges. As odds are operationalized by proportions of wedges, the geometrical probability concept may be applied, whereas by equal sized wedges the problem may be solved via Laplace probability.

We define the concept of cognitive strategy when referring to the game theoretical strategy concept (for games in extensive from, cf. OWEN, 1972). A cognitive strategy is a complete decision plan which generates a decision through the use of knowledge elements, heuristics, goals, and evaluative operators for each state or situation within the cognitive system (cf. SCHOLZ, 1986).

Two–spinner tasks with equal sized wedges may be represented by proportions. Hence the development of cognitive abilities in two–spinner tasks is closely linked to the development of proportional reasoning. We will briefly sketch two approaches to a conceptualization of cognitive strategies in this domain.

The basis assumption underlying SIEGLER's (1981) so–called rule assessment approach is that cognitive development can be characterized as the acquisition of increasingly powerful rules for solving problems. The developmental stage, and hence the repertoire of strategies that may be used at a certain stage, is determined by the degree of complexity of the rules. Within an application of his approach to an urn like probability apparatus, he distinguishes between a dominant dimension, by gambling tasks the favorable events, and a subordinate dimension, that are the unfavorable events. The subsequent rules define strategies for two–spinner tasks within the above meaning:

1. Only the dominant dimension is considered, the urn or disk with a higher number of favorable events is choosen.
2. Rule I is applied, however by an equal number of favorable and unfavorable events the urn (disk) with the smaller number of unfavorable events is selected.
3. The difference between the number of favorable and unfavorable events is calculated for each urn (disk) and the one with the greater difference is selected.
4. The ratio between favorable and unfavorable events is the choice criterion.

Another classical stage theoretical approach for proportional reasoning, which may be applied to two–spinner tasks, is provided by NOELTING (1980). In his experiments he used the so–called orange–juice–paradigm in which the task consists of selecting the mixture with the more intense orange taste out of two mixtures of water and orange juice concentrate. Following Piagets terminology and denoting a mixture with J parts of juice and W parts of water by (J, W), the developmental stages are labeled as follows:

Stage	Name	Characteristics	typical item
0	Symbolic	Identification of elements	(1, 0) vs (0, 1)
I	Intuitive	IA, low; comparison of first terms	(4, 1) vs (1, 4)
		IB, middle; equal first terms, comparison of second terms	(1, 2) vs (1, 5)
		IC, higher; more versus less	(3, 4) vs (2, 1)

II	Concrete operational	lower; equivalence class of (1, 1)	(1, 1) vs (2, 2)
		higher; equivalence class of any ratio	(2, 3) vs (4, 6)
III	Formal operational	low; ratios with one pair of corresponding term multiples of one another	(1, 3) vs (2, 5) or (2, 3) vs (3, 6)
		high; any ratio	(4, 5) vs (5, 6)

The objective of the study. When analyzing childrens' choice behavior in two-spinner tasks we wish to demonstrate that the strategy space is more encompassing than the introduced theories propose. In particular, we want to prove that even children aged between 5 and 10 are able to solve tasks which according to SIEGLER and NOELTING may only be solved in the highest stage via the application of various auxiliary strategies. By auxiliary strategies we mean strategies that are applied instead of the formal-operational strategies which result in the normative solution in all tasks.

Perceptual auxiliary strategies. In many tasks in which the geometrical probability concept may be applied (such as urn or spinner tasks) optical stimuli are present. In such tasks, we call a strategy a perceptual auxiliary strategy if the choice is based on the intensity of the visual stimuli and the direct perceptual impressions. We suppose that young children who are not able to multiply or to calculate ratios may cope with tasks and produce a significant number of correct responses even in NOELTINGs stage III type tasks. In particular we expect more normative responses when the odds difference between the disks increases.

Fifty-fifty comparisons. Many definitions of probability include the equally likely concept as a nondefined or circularily defined concept. We suppose that the equally likely or the fifty-fifty concept is used within reasoning processes on two-spinner tasks. Disks will be denoted as favorable (F), neutral (N), or unfavorable (U), if the chance for a gain is above, equal, or less than .5. If the order of representation is ignored, one may construct tasks for the following five favorability-combinations: F vs F, F vs N, F vs U, U vs U, and U vs N. We hypothesize that the fifty-fifty concept is an element of the individual's strategy space which is available early in the course of development. As a consequence we presume that tasks which contain one neutral disk will more frequently be solved correctly than a task in which both disks are of the same favorability type. Clearly tasks of the F vs U type may also be solved by either applying the equally likely concept as a comparison measure or by directly referring to the more vs less concept. Hence we suppose that tasks of the latter favorability type provide the lowest error rate.

Multivariate analysis for strategy exploration. Besides testing the effects of the independent variables as an indicator of certain strategies we will investigate the strategies which were used with an exploratory cluster analysis.

EXPERIMENT

Subjects. – The subjects numbered 1 to 10 were preschool (M = 5, 10 years), 11 to 20 third formers (M = 9, 4 years), and 21 to 30 seventh formers (M = 13, 10 years). For each age level, the sex was balanced and the heterogeneity of social status, achievement and school type was controlled as far as possible.

Design and rationale. – Twenty two-spinner tasks were presented. On each spinner disk there were equal-sized wedges of red and blue which were separated by lines. The winning color was determined and fixed for each twenty trial run.

There were five introductory tasks followed by 15 tasks that were subsequently analyzed. These disk pairs were constructed in such a way that the number of favorable and unfavorable wedges provided conflicting decisions. For each disk, the number of favorable and unfavorable events were relatively prime. Thus according to the above theories, by one of the two winning colors the task could only be solved at the highest developmental stage. For each favorability combination three pairs were constructed with the odds differences d1 = .05, d2 = .10, and d3 = .20. The odds variable, the favorability combination, and the age group constitute the experimental design.

Table 1: Item matrix ordered according to odds difference and favorability type

(4, 7) vs (3, 4)	(3, 7) vs (2, 3)	(3, 11) vs (2, 3)
(4, 5) vs (2, 2)	(5, 7) vs (2, 2)	(3, 7) vs (2, 2)
(8, 9) vs (7, 6)	(5, 6) vs (4, 3)	(5, 7) vs (3, 2)
(6, 7) vs (2, 3)	(5, 8) vs (2, 5)	(7, 8) vs (2, 5)
(5, 4) vs (2, 2)	(5, 5) vs (2, 2)	(7, 3) vs (7, 3)

Apparatus and procedure. The data were collected in individual sessions at Bielefeld University. Each Subject participated in a game in which a toy figure had to climb a ladder. The Subjects had to choose between a blue and a red figure the color of which determined the winning color in the first run of a roulette-like game. In the second run, which took place on another day, the winning color was reversed. In each trial of a run the Experimenter displayed two disks. The Subject was asked to choose the more favorable disk which was then inserted into a spinning wheel which was usually spun by the Subject. If the outcome was the winning color, the toy figure was moved upwards, otherwise it remained on its place. When the fifth step was reached, the subjects received gains (i.e. a choice between sweets or money). After the five introductory tasks the above items were presented in a random order.

RESULTS

As can be seen from Table 2, the age variable yielded a significant main effect upon the error rate ($H = 21.6$, $p < .01$), but the odds variable also produced significant effects (d1: $H = 15.3$, $p < .01$, d2: $H = 18.1$, $p < .01$; d3: $H = 14.7$, $p < .01$).

Table 2: Percentages of normative correct solutions seperated for age and odds difference

	Age level			
Odds Difference	5 - 6	9 - 10	13 - 14	all
d1 = .05	54	64	85	67.6
d2 = .10	61	69	95	75.0
d2 = .20	66	91	97	85.6
all tasks	60.3	74.6	92.3	75.1

The odds difference hypothesis could also be confirmed. An appropriate test is provided by KENDALL's tau (computed via the ranks of the individuals' solution frequencies for the different difference levels). Across all Subjects and all groups the effect was highly significant. In agreement with our theoretical considerations, the preschool children's error rate was above chance level even for tasks of the highest stage that may only be solved via ratio comparisons or multiplication.

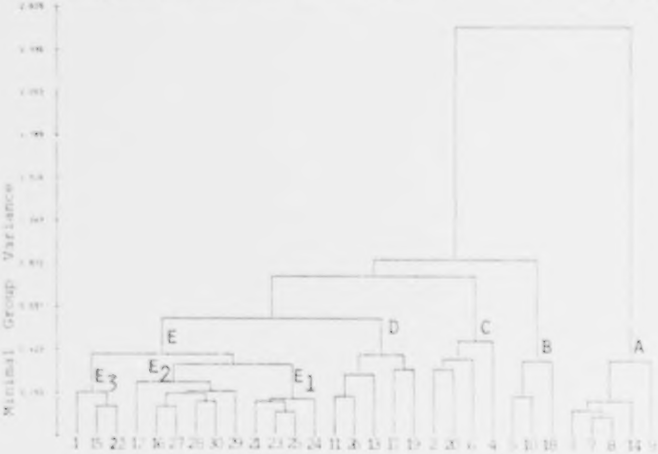
Table 3: Percentage of normative solutions seperated by age and favorability combination

	Age level			
Favorability combination	5 - 6	9 - 10	13 - 14	all Subjects
A vs A or D vs D	57.5	72.2	88.3	72.8
A vs N or D vs N	60.8	76.6	95.0	77.5
G vs D	65.0	75.0	95.0	78.3

Tasks with the same favorability type for both disks show the highest error rate. Across all Subjects KENDALL's tau indicated a significant trend through the different favorability combinations ($p < .001$), whereas we want to note that for the preschool children this trend was only of marginal significance ($p < .10$) and that there was no significant effect if only the third graders are considered.

Clusteranalysis for strategy identification. Clusteranalyses are descriptive methods for grouping multivariate attribute vectors. We supplied this method to the Subjects solution vectors. Subjects of a certain cluster thus show similarity with respect to their response behavior. Figure 1 presents the dendrogram of a hierarchical cluster analysis. Based on the euclidian distance measure, **WARD's – Method** was applied as an agglomeration procedure with five interpretable clusters to extract (cf. **WISHART, 1978**). These clusters are labelled from the left to the right by A, B, C, D, and E and contain the Subjects with the numbers at the base of the tree. We will introduce two further specific strategies, first the so – called counter strategy in which the decision is based on the absolute number of favorable events, and the denominator strategy, in which the decision is only based on the absolute number of unfavorable events.

Figure 1: Dendrogram of solution vector clustering, y – axis minimal values of group variance



The identification of the cluster characteristics is based on the following parameters: (1) all over relative error frequency (x), (2) error frequency for all items which may be solved correctly by comparing counter (xc) or denominator (xd) strategy, (3) error frequency for tasks with odds difference of d1 (xd1), d2 (xd2), or d^3 (xd^3), and (4) error frequency for tasks with favorability combination F vs F or U vs U (tied to xFF), F vs N or U vs N (tied to xN), and F vs U (xFU). In addition, specifities or the clusters' solution vectors are controlled in an exploratory manner.

Cluster A: In terms of cluster analysis, this cluster in which preschoolers predominate is an extreme one which differs strongly from other clusters shows the highest error rate x = .47, and its members clearly exclusively used the counter strategy, as a value of xc = .04 is contrasted with a value of xd = .91.

Cluster B: This cluster shows an intriguing characteristic. For the counter tasks (i.e. tasks in which the counterstrategy yields the correct solution) it presents an approximately equal to slightly raised error rate xc = .38 vs. xd = .33, showing a minor improvement with increasing odds differences, and a marked effect of favorablity combination, xFF = .50, xN = .37, xFU = .08, particularly in the cluster's core.

Clusters C and D are relatively heterogenous clusters in which we were unable to recognize any marked and consistent structural similarities. Also, these clusters proved to be unstable, because they dissolve, if other methods of agglomeration (such as the group average method) are applied. Hence, we shall skip their detailed description and merely note that the Cluster C in which preschool kids dominate is marked by a high number of errors for counter tasks (xc = .32) and denominator tasks (xd = .40), while Cluster D in which third formers dominate is marked by a relatively high number of errors for counter tasks (xc = .32), a relatively small error rate for denominator tasks (xd = .14), and an evident mastery of tasks with the highest odds difference (xd = .26, xd2 = .36, and xd3 = .06).

Cluster E: This cluster contains a total of 13 subjects, including all but one of the seventh grades. Thus it is not surprising that this cluster shows an error percentage of only x = .10. Its other characteristics are similar to those of the overall population, with the exception of the performance in counter and denominator strategies. Here, this group commits approximately an equal number of errors for the two types of tasks (xc = .09 vs. xd = .10), while these two values are markedly different for the total population (xc = .18 and xd = .31).

Within Cluster E, there are three (relatively stable) subclusters that show some differences in strategy. They will be briefly described, despite the fact that they lend themselves only to a restricted interpretation.

Cluster E1 contains the best of all subjects with x = .06. These subjects made errors almost exclusively in denominator tasks; xc = .01 vs. xd = .10, and no errors occured in tasks of the more – less type. (xFU = .00).

Cluster E2 provided x = .11 errors, whereas no errors occured in tasks with a high odds difference. Curiously enough, this cluster features by a higher error rate in tasks with an easier favorability type, xFF = .08, xN = .10, and xFU = .17.

Cluster E3 with x = .14 shows performance by increasing odds difference, whereas the error rate dropped strongly by favorability combination; xFF = .25, xN = .08, and xFU = .00.

INTERPRETATIONS AND CONCLUSIONS

In our opinion, the results permit the following interpretations:

– Perceptive auxiliar strategies are obviously already applied by preschoolers, and lead to a significant rate of normatively correct answers by two – spinner roulette tasks. Using the terminology of FISCHBEIN, 1975, and SCHOLZ, 1986, these auxiliar strategies represent intuitive strategies of stochastic thinking.

– By equal formal – operative difficulty and equal odds differences, tasks in which a favorable or unfavorable roulette disk is compared to a "neutral" one, or tasks in which one disk is favorable and the other unfavorable, are more frequently solved correctly than tasks in which both disks are either favorable or unfavorable. This is a clear indication of an equally likely concept as a strategy component, a phenomenon we have called fifty – fifty comparison. We should like to interpret this result in the sense, that a conceptualization of both stochastic and proportional thinking must not only take into account, the procedural rule – governed (cf. SIEGLER, 1981), and the operative aspect (cf. NOELTING, 1980), but also consider the semantical – conceptual aspect (cf. SCHOLZ & WALLER, 1983), as well as the structual, contextual of the task features.

It is well known that formally operative heuristics or analytical strategies are not generally superior to intuitive strategies (as operationally defined by SCHOLZ, 1986). Thus, it may seem almost trivial that the so – called counter strategy is, by certain tasks, inferior to the visual – perceptive auxiliary strategy identified in this paper.

Cluster analysis served to identify groups of subjects who showed typical patterns of strategies.

Cluster A consisted predominantly of preschoolers who resorted to just one strategy, the counter strategy. Another cluster which predominantly contains younger subjects is apparently not oriented toward the number of winning wedges, but is marked, by the use of perceptive auxiliary strategies, and the ability to solve more – less tasks.

For one group of subjects (Cluster D), the analysis of the solution vectors indicated that the subjects used a variety of strategies in a more or less systematic way. This cluster contains half of the third formers. The error rate of 22 percent reveals the partial success but also the uncertainties connected with the change of strategy.

Cluster E, which mainly consists of seventh formers, contains those subjects who show mastery of the tasks presented. The characteristics of visual and conceptual – numerical orientation partially developed in Clusters B and C can also be found in this cluster. Thus, the high rate of success in Cluster E2 is probably due to an elaborated, highly sensitive perceptive strategy which is almost error – free by odds differences above either .10. Conceptual components operating with the equally likely or the more – less concept are apparently unimportant. Cluster E3, however, seems to be strongly dependent on conceptual – numerical patterns, as the counter tasks and denominator tasks have been processed with noticeably different success rates and as a trend towards the favorability combination can be identified.

These results reveal, on the one hand, that applying SIEGLER's or NOELTING's theories to two – disk roulette tasks seems to require the addition of a visual component of strategy, and, on the other hand, that comparisons using the equally likely concept play a part in development.

The problem of the flexible use of auxiliary strategies or of other strategies is by no means trivial from an educational or developmental point of view, particular as this problem is linked, in the field of stochastic thinking, to the difficult problems of the various foundations or facettes of the concept of probability. Geometrical probabilities and perceptual auxiliary strategies which may be appropriate for two – spinner tasks, can be systematically misleading in other tasks.

To close, we should like to note that the strategies described can also be identified in content analyses of thinking aloud protocols. The study of protocols, however, also yields that the list of auxiliary strategies is by no means exhausted by those represented in this paper. A further strategy typically used by several individuals is the strategy of preferring the disk with the larger winning sectors; and another perceptually operative strategy is that using reasoning processes to visually form blocks of winning and losing sectors. Our further investigations will be aimed at listing and formulating these strategies within the terms of the framework for the structure and processing of stochastic decision making as sketched by SCHOLZ (1986).

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STRATEGIES CHILDREN USE IN SOLVING PROBLEMS

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Results in an earlier project led to the question, How do children choose the arithmetic operations they use in solving routine story problems in mathematics? One would naturally hope to find considerable evidence that choices depend on the children's having meanings for the operations—e.g., that one can use multiplication when several sets, each with the same number, are put together—and that these meanings are used to model the settings of story problems. Such an approach might be called a concept-driven strategy.

BACKGROUND

Results on nation-wide testings in the U.S.A. have led commentators to note that, while performance on one-step story problems may be acceptable, performance on multistep problems or problems involving extraneous data is relatively poor (Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1980). Thus one might conclude that children in the U.S.A. have meanings for the operations but have difficulty in matching their schemata for the operations to settings more complicated than those of the usual one-step problem.

Several recent European studies have noted the surprisingly weak performance of learners on story problems using decimals and multiplication or division (e.g., Bell, Fischbein, & Greer, 1984; Bell, Swan, & Taylor, 1981; Ekenstam & Greger, 1983; Fischbein, Deri, Nello, & Marino, 1985; Greer & Mangan, 1984; Greer, to appear).

Apparently the unintended curriculum with whole numbers establishes so well the false-in-general "multiplication makes bigger, division makes smaller" that a change from whole numbers to decimals less than one often results in the learner's choosing a different operation to answer the same question in the same context. This "nonconservation of operation" (to use Greer's label) is symptomatic of a failure to use a meaning for multiplication or division in at least one of the problems and, perhaps, in both.

Earlier paper-and-pencil testings had suggested that learners did not have concepts for operations (Sowder, 1984). The test of that study has dubious validity however, depending as it does on simple story problems. Although the limitations of interview studies are well-known, only interviews seemed capable of determining whether learners do have concepts for the operations and whether these concepts are the basis for the learners' choices of operations in solving story problems.

THE INTERVIEW STUDY

Group tests of story problems, most recently in grades 6 and 7 but earlier in grades 4-8, were used to identify learners for individual interviews. Learners with at least average group test performance were selected, with teacher input, for the interviews. Selected group-test items, with additional problems, served as the basis for each interview. The recent focus has been on multiplication and division situations. In American curricula, students in grades 6 and 7 have been exposed to multiplication and division since grade 3.

THE STRATEGIES

Following are the strategies that have been observed. The excerpts from selected interviews give the flavor of the dialogue judged to signal the strategy.

Strategy 1 Find the numbers and add.

Strategy 2 Guess at the operation to be used.

Strategy 3 Look at the numbers; they will "tell" you what operation to use.

Example: (Grade 7 student. S = student; I = interviewer)
S: "Yeah, it looks like a division....It's the numbers, I guess."
I: "Oh, the numbers, huh?"
S: "3 times, 3 goes into 78, that's what it mostly is. Cause if it's like, 78 and maybe 54, then I'd probably either add or multiply. But 3, it looks like a division. Because of the size of the numbers."

Strategy 4 Try all, +, −, x, ÷, and choose the answer that is most reasonable.

Example 1: (Grade 6 student, about a problem involving x)
I: (After S discusses rejecting + for the problem) "Did you even think about adding, a couple of weeks ago (during a group test)?"
S: "Yeah. I go through every one to see if it would work."
I: "Oh, you do?..."
S: "And I went through adding, and I saw that that wasn't a good choice, and then I went into subtraction..."

Example 2: (Grade 6 student)
I: "...what made you think you should divide?"
S: "Well, the addition, subtracting, and multiplying didn't look right."

Strategy 5 Look for isolated "key" words to tell what operation to use. (E.g., "all together" would mean add, "left" would mean subtract, "of" would mean multiply.)

Strategy 6 Decide whether the answer should be larger or smaller than the given numbers. If larger, try both + and x, and choose the more reasonable answer. If smaller, try both − and ÷, and choose the more reasonable.

Example 1: (Grade 7 student)
S: "...that (problem) was adding, multiplying. I didn't even bother with them, subtracting or dividing."

Example 2: (Grade 6 student)
S: "Well, I would think that you have to subtract, because, er, it'd either be a subtract or, um, division, and then the one that sounds right would be the subtraction..."

Strategy 7 Choose the operation whose meaning fits the story.

Example 1: (Grade 6 student)
S: "...There's 24 in each row, and there's 12 rows. And you want to know how many there were, so you'd times."

Example 2: (Grade 7 student)
S: "...I'd probably multiply it...I would multiply it but you could also add it. But it would take you, you would have to add 2.46 three times."

The last strategy is the desired strategy, of course, but it was rarely evident. Even apparently capable learners had difficulty articulating their reasons for choosing a particular operation. Zweng (1979) has pointed out that curricula in the U.S.A. typically do not provide the learners with vocabulary with which to link operations and story problem situations.

IMPLICATIONS

It should be noted that many of the immature strategies do give correct choices of operation for many one-step story problems. Hence, the apparently satisfactory performance on one-step problems in such testings as the National Assessment of Educational Progress in the U.S.A. is put under a cloud. If students are succeeding primarily by use of the immature strategies, the results are tainted. Indeed, an indirect confirmation of this possibility is the far worse performance on multistep story problems, for which the immature strategies are likely to break down.

Use of Strategy 6 may complement the incomplete "multiplication makes bigger, division makes smaller" as the approach used by many nonconservers of operation. Multiplication might be correctly chosen for a whole number problem not because it models the situation but because it gives an answer of the correct magnitude, and multiplication might be rejected for a similar problem with a decimal less than one because the anticipated product would not be small enough. The anticipated size of the answer, not any meaning for an operation, may drive the choice of operation in each problem.

If the immature strategies are indeed in common use, the instructional implications are clear: They must be discouraged, perhaps by more frequent use of multistep problems or problems with extraneous information, or better yet, supplanted with concept-driven strategies.

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Calculators and Realistic Arithmetic Instruction
Developing the calculating automaton

Jan van den Brink

I. *Three kinds of knowledge around calculators*

a. *Arithmetical knowledge is necessary* for an exploration of the calculator that eventually leads to knowledge on calculators.

b. *Knowledge about (various) calculators is necessary* for acquiring a mental object of calculator _ a mental object which (such as that of number) even increases with the variety of calculators available.

This knowledge of calculators includes general and specific ways of operating.

c. *Knowledge about applying* calculators in context situations is the next step. Besides knowledge about operating it includes a kind of 'action algebra', for instance knowing how to figure out 3% of f 27.- while using the calculator _ there are several ways to do it.

Only the latter instructional situation is usually experienced as typically 'realistic' because *contexts* are explicitely involved. But the abilities a and b fit as well into the realistic framework (Treffers & Goffree, 1985), since calculators represent part of a reality worth exploring.

II. *Knowledge of calculators*

Though seemingly in contradiction to common thought calculators in action can exhibit strange discrepancies of behaviour. To 4 x 5 - 4 x 5 = one calculator answers 80 whereas the other says 0.

Children don't care. The one calculator works straight linearly whereas the other respects the Algebraic Operating System (AOS). Both answers are *accepted*. Not before the end of the first lesson did the sixth-graders become wary _ the start of a crisis. The problem arose when *bare* arithmetic was put *into a context*.

4 x 5 - 4 x 5 =

Mohammed (6th grade): 'That is nought. One can hear it.'

But surprisingly his calculator showed 80. Thanks to the *instructional context* he had expected something else.

Michael (6th grade): 'Isn't it a shame, Mister, these have different results?'

I: 'Why?'

M.: You never know what is correct. The teacher's calculator can give another result. And then you get a red mark

Lateron the *children themselves* discover various remedies against the AOS illness:

o splitting the calculator process,
o inserting somewhere an equality sign
 (4 x 5 - 4 = x 5 = 80 is unambiguous)
o using parentheses
o using the memory

It should be emphasised that the children *themselves* felt the *necessity* to find out such remedies _ necessity as a *source* of mathematising

3 x 3 x 3 = 81 or 9
Another aberration of the calculator due to the speed of pressing keys
For instance, with Casio -∫ x 80 it happens that 3 x 3 x 3 pressed slowly gives 27
But done hastily, it becomes 81, and at full speed, it can be 9

Aberrations, diagnosis and remedy
There are a host of aberrations, which usually are discounted as side-effects
Indeed they derange calculations
However, a few striking facts are worth mentioning:

o rather than from adults I learnt this kind of aberrations from *children*. Some children keep their knowledge secret lest they should buy another calculator.

o If one calculator deviates from another one or from itself or from what the user expects, the first reaction is to ask *What is wrong? Which are the causes?*

Then they try to 'diagnose' the 'disease'.
Example:
How comes 3 x 3 x 3 = 81 or 9?
The calculator did not read
* the last 3 of 3 x 3 x 3 =
* the second 3
* the equality sign in 3 x 3 x 3 =
Or:
* The reading and the calculating department are not in step with each other.
Most often several diagnoses are proposed.
In order to find the true cause, so-called *'differential diagnosis'* is needed. By changing the numbers involved some possible causes can be eliminated:
Example:

3 x 3 x 3 = (quick) 81 or (rapid) 9.
3 x 3 x = also gives 81 _ the last 3 being dropped such as in
2 x 3 x 5 = (quick) 36 _ the 5 being dropped.
3 x x 3 = gives 9 _ the second 3 being dropped such as in
3 x 6 x 3 = (rapid) 9 _ the 6 being dropped.

We noticed that
- Calculator aberrations are better known by children than by adults
- the aberrations illicit looking for causes and remedies.

Syndromes
We display a few syndromes, which often occur if children use calculators

a. the pressing syndrome

o slow or lasting pressing produces $\boxed{333333.}$ on the display,

o soft pressing produces $\boxed{0.}$

b. the stress syndrome

$3 \times 3 \times 3 =$ slowly pressed, produces 27.

$3 \times 3 \times 3 =$ quickly pressed, produces 81.

$3 \times 3 \times 3 =$ rapidly pressed, produces 9.

c. the confinement syndrome

$50000000 \times 2 = \boxed{1.}$ or $\boxed{1.0000000}$ or $\boxed{1.\ E\,8}$

(seven noughts)

or even $\boxed{100.}$ if $50.000.000 \times 2 =$ is pressed.

(Notice that in continental Europe the role of the decimal point and comma are interchanged)

The calculators tend to confine numbers to the display, which causes strange results if large numbers are concerned.

d. the form syndrome

Example:

the long division form 109/11009\ gives 101but it can also be 0.0099009 (by way of 109 ÷ 11009=).

e. the big shot syndrome.

Parts of small numbers, right of the decimal point (for instance) might succumb to the occurrence of large numbers.

Example:

$0.3333333 + 4444444 - 4444444 \neq 0.3333333$

f. the persistence syndrome

Former commands may persist.

Example

$1 : 2 = \boxed{0.5}$ arc sin $\boxed{30} = \boxed{15}$.

The result 30 is divided by 2 as though there were no 'arc sin'.

In particular if trigonometric functions are involved one should be careful to distinguish argument and its function value.

g. the A O S syndrome

Combining the various arithmetical operations can cause problems.
Examples:

$4 \times 5 - 4 \times 5 = \boxed{0}$ as well as $\boxed{80}$

$1 \times 3 = \div 3 =$ and $1 \div 3 = \times 3 =$

can yield different results.

III *Instructional Practice*

The approach towards calculators as (aberrant) micro worlds, which can be explored and compared, influences *instructional practice*.

There are many ways to use calculators in this sense in mathematics instruction.

* having *manuals* written by the pupils for their kind of calculator in order to register their discoveries ('pupils' own productions').

* stimulating *chains of expertise* in the class room ('interactive learning from each other')

* *collective class investigation* about the use of calculators (starting from real phenomena; developing interactivity),

* which can result in *syndrome descriptions, etiology,* and *remediation* of calculators — children as calculator physicians on a medical board. (Tools of vertical mathematisation are available: A O S, arrows language).

* design, exchange and negotiation of *calculator problems,* that is, of the kind of sums that yield different results on different calculators ('interactivity', 'negotiations').
Notice: Diverging results of equal procedures are motivating investigation more than do equal results by different procedures.

* making good use of the opportunity offered by calculators with a view on *transfer* of procedures to each other is investigated (Pimm 1981) (horizontal mathematisation).
One calculator can sit as a model for other ones.
(Cp. the use of contexts as models, Treffers & de Moor 1984)

* Investigating *calculator bound numbers and operations* such as the scientific notation, products by partial multiplications, and so on.

Much more can be said on the consequences for instruction, both with regard to subject matter contents and ways of teaching, but this would lead us too far away.

IV *Instruction Theory Remarks*

While taking in mind a course on peculiarities of various kinds of calculators, I feel inclined to reflect on it in terms of instruction theory and to propose some theses of global character as well as ones on micro level. Let us start with the last kind.

1. Each calculator represents a *piece of reality,* a *micro-world,* a *context,* which should be investigated and compared with similar ones (notations, operational rules).
2. When comparing calculators each pupil uses his own calculator as a *model* in order to investigate the other ones — points of agreement and of divergence.
   It looks a bit like the use of models as happens if the buses model is used in other contexts.
3. This involves mathematising both in the *horizontal* (various calculators) and in the

vertical sense. Knowledge about one's own calculator extends to that on other ones (horizontally). First the children knew about the existence of differences, then they learn about their character (keys, rules). They placidly accepted the differences because they had to. The development is similar to that of the number concept, which presents itself in different mental objects (numerosity, ordinal number, number concept of the Papuas, and so on). Similarily there are various kinds of calculators.

4. There are, however, striking contrasts between the development of the number concept and the calculator's one:
   o The number concept develops *gradually*.
   Moreover it is a means of *adult* description rather than a subject of use by the learner.
   o The need for developing a general understanding of calculators is *suddenly* experienced *by the children*: calculators are reckoning differently, which should not be allowed. The compulsory school contexts (ordinary arithmetic, social rules in the classroom) effectuate a *crisis* leading to vertical mathematisation.
5. By the pupils the development of a general calculator concept (the *calculating automaton* in the sense of the theory of automata) is experienced as a necessity. As an idea it is not vague since it is practically elaborated by letting different calculators work in the same way.
6. Summarisingly we can distinguish four phases in the course we have in mind:

   A. *isolation*
   No attention is (yet) paid to *differences* between the computing of the calculator and ordinary arithmetic.

   B. *acceptation*
   The various computing methods of calculators are identified and accepted as equally justified: they are mere procedures among which there is no preference. One is not astonished. This phase can last quite long.

   C. *crisis or confrontation*.
   In order to overcome the acceptation of various procedures, calculators must be placed into a context.
   Only in a context can the crisis of contrast between two bare calculating methods arise.

   D. *restoration*
   Means are created to have all calculators in question calculating according to the children's wishes.
   By this way a logical systematic was developed by the children.

7. We refrain from tackling more consequences of considering calculators as mental objects representing the calculating automaton. They are related to questions like the following ones:
   Is the calculator really a didactical aid?
   Isn't the choice of calculating activities determined by the context?
   In which way does calculating depend on material bound properties?

Finally we consider the development towards the calculating automaton in terms of instruction theories like Van Hiele's (1957) and Treffers' (Treffers and Goffree 1985).

8. The course we have in mind about various calculators can illustrate the Van Hiele levels.

   At level 1 subjective sensory perceptible objects are investigated, in the present case the various calculators with their specific peculiarities.

   At level 2 the *relations* between them are focused on: the one calculating linearly from the left to the right, the other according to AOS; there are corresponding differences of notation.

   At level 3 the *logical/systematic* connection is at issue: the children discover all kind of means to manipulate the calculators and in this way create the first mental object of the calculating automaton.

   One can also distinguish the five phases leading from one level to the next: Information (via non-curricular calculators), bound orientation (per calculator, with various calculators), explicitation (by emphasising the divergence and finding solutions), free orientation (by exchanging calculator problems), integration by extending regular arithmetic concepts like positional system, decimal numbers, multiplying).

9. Treffers' five tenets of realistic arithmetic instruction according to Wiskobas apply:
   1. Start from real phenomena (various calculators in action).
   2. Means of vertical mathematisation are made available by specific arithmetical knowledge (calculating according to AOS; arrows language; consulting each other).
   3. Large contributions made by the children themselves: each has his own calculator and investigates the other ones with his own as a model.
   4. Interactivity is both compulsory (by need for finding solutions of the problems) and optional (in the transaction with calculator problems).
   5. Learning strands are intertwined: the calculators represent numbers and operations in a strangely extending way (scientific notation and decimals, multiplying both as iterated adding and straightforwardly).

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